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A New Sequential Goodness of Fit Test for the Three-Parameter Weibull Distribution with Known Shape Based on Skewness and Kurtosis

THESIS

Jonathan C. Clough
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Jonathan C. Clough

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Abstract

The Weibull distribution finds wide applicability across a broad spectrum of disciplines and is very prevalent in reliability theory. Consequently, numerous statistical tests have been developed to determine whether sample data can be adequately modeled with this distribution. Unfortunately, the majority of these goodness-of-fit tests involve a substantial degree of computational complexity. The study presented here develops and evaluates a new sequential goodness-of-fit test for the three-parameter Weibull distribution with a known shape that delivers power comparable to popular procedures while dramatically reducing computational requirements. The new procedure consists of two distinct tests, using only the sample skewness and sample kurtosis as test statistics. Critical values are derived using large Monte Carlo simulations for known shapes $\beta = 0.5(0.5)4$ and sample sizes $n = 5(5)50$. Attained significance levels for all combinations of the two tests between $\alpha = 0.01(0.01)0.20$ are also approximated with Monte Carlo simulations and presented in a simplified graphical format. Extensive power studies against numerous alternate distributions demonstrate the test's excellent performance compared to popular EDF test statistics such as the Anderson-Darling and Cramér-von Mises tests. Recommendations are included on techniques to choose significance levels of the two component tests in a manner that should optimize power while maintaining the overall significance level.

A New Sequential Goodness of Fit Test for the Three-Parameter Weibull Distribution with Known Shape Based on Skewness and Kurtosis

I. INTRODUCTION

1.1 Background

The utility of reliability theory and life-testing models in the development, procurement and improvement of advanced systems for the USAF has been and is vitally important, particularly in the current era of fiscal constraint and budget reductions. Modern systems are often so complex that purely analytical approaches to understanding their operation are insufficient. Instead, simulation and modeling are often required to assess their reliability. While simulation is a tremendous tool, the input data for simulations are critical for model accuracy. In a typical reliability problem, time-to-failure data of individual components of a complex system are key simulation inputs. These input data are the "driving force" for a simulation model [9: 355], and improper representation of that data will lead to misleading results, which could have a severely detrimental impact on a particular program if key decisions are based on model results. Accurate model inputs, then, are of fundamental importance to any such model-based analysis, be it for the USAF or a commercial establishment.

Simulations normally employ probability distributions to model the stochastic nature of input data, but determining the distribution to approximate empirical data is often a daunting task, particularly when there is limited data available. Goodness-of-fit tests address this problem and are used commonly to ensure model inputs are consistent with observed data. The general model development procedure leading to the application of a goodness-of-fit test follows four basic steps:

- collect data on the system components being analyzed (e.g. time to failure data)
- identify or hypothesize a distribution or family of distributions to represent the data

- estimate the parameters of the hypothesized distribution
- conduct a test to determine whether or not the hypothesized distribution effectively and accurately represents the data [9: 355-56].

An extensive body of goodness-of fit tests have been developed for all types of distributional inputs. Most tests are developed for use in testing particular families of distributions, and their performance varies based on what alternatives are being weighed against the hypothesis. In selecting a test, the analyst will opt for the one that will offer him the highest power, the probability of correctly rejecting the hypothesis when it is indeed false [71: 455]. A test with a high power for a particular distribution will properly discriminate between that distribution and a range of alternatives. The goal for the analyst in this effort is to identify tests with the highest power for the type of data he seeks to analyze. If the test accepts the data, the hypothesized distribution can be employed as an input to a simulation, for example, that might compute vital reliability characteristics of a given system (e.g. shelf life, time to failure, etc.)

In life-testing and reliability applications, the Weibull distribution is one of the most widely used distributions, second only to the exponential [43: 291] [1: 59] [6: 590]. Mann has listed numerous examples of its applications in failure analysis [47: 185]. Its utility, however, is not limited to the field of reliability; it has been used extensively in other fields as well [81]. First introduced by Swedish engineer Waloddi Weibull in 1939 to characterize the breaking strength of materials, the distribution has found widespread applicability [72]. Weibull himself demonstrated its utility in describing phenomenon diverse as the fatigue of steel to the fiber strength of cotton [73]. More recently, the distribution has proven useful in discriminating between man-made and natural structures in imagery collected by synthetic aperture radar (SAR) [14], describing the distribution of particle sizes for binary Lithium-Aluminum alloys [30], and modeling radar clutter to improve detection capability [50]. The Weibull distribution clearly has demonstrated utility in many disparate fields of study, including reliability analysis. Due to its widespread application and flexibility, efforts to

accurately identify data that can be modeled by the Weibull are valuable to a broad spectrum of technical endeavors. Consequently, effective goodness-of-fit tests for the Weibull distribution are critical to the validity of these applications [78: 133].

1.2 Objective:

The objective of this research effort will be to develop, implement, and analyze a new goodness-of-fit test for a particular family of the Weibull distribution in which the shape parameter is known, but the location and scale parameters are unknown. The test will utilize the third and fourth standardized sample moments of the data, known as sample skewness and sample kurtosis, to evaluate the goodness-of-fit for the Weibull distribution. The procedure will actually consist of a sequential test, in which one test (using sample skewness) will be conducted followed by a second test (using kurtosis); a sample fails the sequential test if it fails just one of the two tests. It is hoped that the sequential application of these tests will facilitate discrimination against a wider variety of alternate distributions with consistently equal or higher power than existing tests provide. Such a test is known as an omnibus test [42: 438]. To assess the effectiveness of the test, a power study will be performed to compare the test's performance with current powerful goodness-of-fit tests against a wide variety of alternate distributions.

II. LITERATURE REVIEW

2.1 Weibull Distribution

The Weibull distribution is a widely applicable distribution that belongs to the family of generalized gamma distributions, and it typically has two parameters, a shape and scale parameter. The two-parameter Weibull cumulative distribution function (CDF) and probability density function (PDF) are given by:

$$\begin{aligned}
\text{CDF:} \quad F(x,\beta,\theta) &= 1 - \exp\left(-\frac{x}{\theta}\right)^{\beta} && x \geq 0, \text{ and} \\
\text{PDF:} \quad f(x,\beta,\theta) &= \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} \exp\left(-\frac{x}{\theta}\right)^{\beta} && x \geq 0.
\end{aligned}$$

The **shape** parameter β, as its name suggests, describes the shape of the Weibull PDF. This parameter is one of the keys to the flexibility of the distribution. When $\beta = 1$, the PDF becomes the exponential PDF, often used to model a constant failure rate in reliability studies. A Weibull with $\beta > 1$ models an increasing failure rate and, when $\beta < 1$, the PDF models a decreasing failure rate. The plot in Figure 2.1 illustrates the variety of forms the Weibull PDF can assume by varying the shape parameter.

This flexibility is one trait that makes the Weibull so useful in reliability theory [81: 54] and for modeling a wide variety of other data as well [11: 803]. In most lifetime estimation problems the shape parameter usually falls in the range $0.5 \leq \beta \leq 3.5$ [38: 310]. For $\beta = 3.6023494$, the Weibull PDF is "almost normal" [19: 27], although Kapur and Lambertson warn that a Weibull with this shape is not a true mathematical approximation to the normal distribution [43: 294]. Thus, the shape parameter is the real driver in determining the form the PDF curve assumes. It allows the Weibull distribution to assume a broad variety of contours, contributing to its utility to such diverse disciplines as noted earlier.

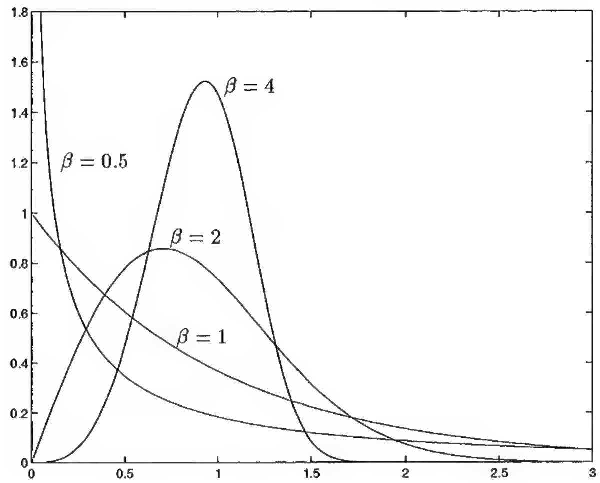


Figure 2.1 Weibull PDF: Scale $\theta = 1$; Shape Varied

The **scale** parameter, θ, describes the dispersion of the random variable about its mean, and is called the characteristic life in reliability applications. It helps to locate the distribution on the x-axis by dividing the area under the PDF curve into the same proportion for all values of β. To illustrate this, observe that the two-parameter Weibull CDF evaluated at $x = \theta$ becomes

$$F(x = \theta) = 1 - e^{-1} \approx 0.632 \quad \text{for all values of } \beta$$

So, the value θ will always divide the area under the PDF into 0.632 and its complement 0.368. Hence, in reliability theory, for a component whose lifetime can be described with a Weibull distribution with scale θ, its probability of failure prior to time θ is always 0.632, thus leading to the characteristic life nomenclature [43: 294].

The plot in Figure 2.2 demonstrates the effect on the Weibull of varying θ for a fixed shape $\beta = 2$. It is evident that as θ increases the probability mass becomes much more dispersed away from its mean. For a given value of the shape parameter, it is obvious that the scale parameter serves to stretch or compress the probability mass about its central tendency.

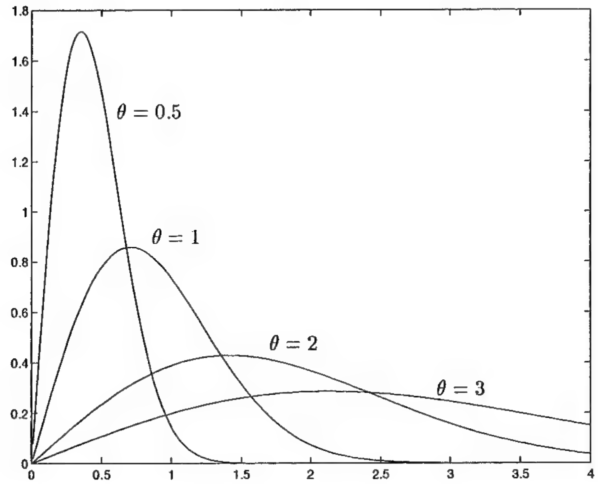


Figure 2.2 Weibull PDF: Shape $\beta = 2$; Scale Varied

In many life-testing applications, failures can occur from the moment a component is put into service; thus, in the two-parameter case, the time to failure $x \geq 0$. However, in other systems a component can fail while in storage, before it begins service; alternately, there are cases where a component may operate failure-free or with very rare failures for some period of time after beginning service [38: 310]. In these scenarios a third parameter δ, a **location** parameter, is introduced. This parameter allows the PDF to be shifted left or right along the x-axis depending on when the earliest failure time can occur, and is referred to as the guaranteed life or minimum life [43: 292]. Adding this third parameter to the Weibull functional form creates the three-parameter Weibull distribution. The CDF and PDF are:

$$\text{CDF:} \quad F(x;\beta,\delta,\theta) = 1 - \exp\left[-\left(\tfrac{x-\delta}{\theta}\right)^{\beta}\right] \qquad \delta \leq x, \quad \text{and}$$

$$\text{PDF:} \quad f(x;\beta,\delta,\theta) = \tfrac{\beta}{\theta}\left(\tfrac{x-\delta}{\theta}\right)^{\beta-1} \exp\left[-\left(\tfrac{x-\delta}{\theta}\right)^{\beta}\right] \qquad \delta \leq x.$$

If $\delta > 0$, the PDF is shifted right, indicating a guaranteed period in which a component is failure free. If $\delta < 0$, the PDF is shifted left, denoting a potential failure in storage (negative failure time). The effect of the location parameter on the distribution is shown in Figure 2.3.

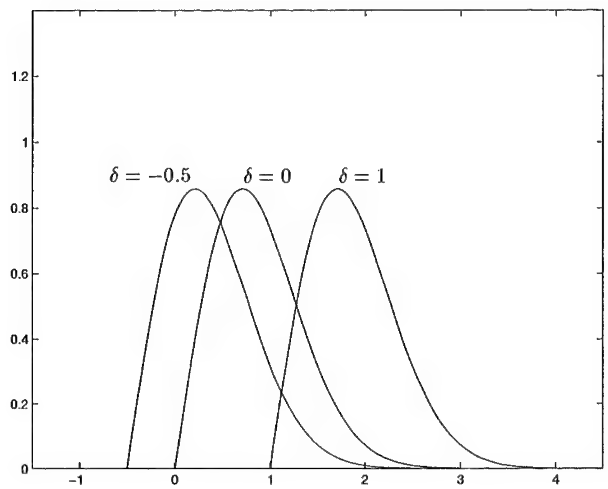


Figure 2.3 Weibull PDF: Shape $\beta = 2$; Scale $\theta = 1$; Location Varied

The mean, μ, and variance, σ^2, of the Weibull distribution are given by:

$$\begin{aligned} \mu &= \theta\Gamma\left(1 + \frac{1}{\beta}\right), \text{ and} \\ \sigma^2 &= \theta^2\left[\Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma^2\left(1 + \frac{1}{\beta}\right)\right]. \end{aligned}$$

As β increases, the mean of the Weibull approaches the characteristic life θ and variance approaches 0 [43: 293].

The Weibull distribution in both its two- and three-parameter forms, is clearly a very flexible distribution suited to describing a broad variety of data. Hager, et al. demonstrated the robustness of the Weibull model in reliability applications compared to the exponential and generalized gamma distributions [32: 547], further complementing the literature noted so far.

2.2 Complete and Censored Data

Because the data used in Weibull goodness-of-fit studies are often associated with life-testing and reliability studies, an understanding of some of the peculiarites in how failure time data are collected is important. In a typical reliability experiment, a set of components are placed on test,

and the time until component failure is observed. The random sample of failure times for the components becomes the basis for inferences on the reliability of the population. Ideally, a sample size n is selected, and the components are observed until all fail, resulting in a sample of n failure times. Data collected in this manner is called a complete sample. In practical applications, however, it is often economically infeasible to wait until the entire set of components fail. As a result, life-testing experiments are usually terminated at some point before all components fail. Samples collected from such experiments are called censored samples – the sample size r of failure-time data is smaller than the number of components placed on test, n.

Censoring is further categorized by the experimental methodology employed to collect the data. In **Type I censoring**, as items fail they are replaced, and the experiment is halted after a fixed amount of time. In this case, the length of the experiment is specified but the number of observations obtained is a random variable. In contrast, **Type II censoring** does not replace failed subjects and terminates the experiment as soon as a fixed number of r observed failures are obtained. Here, the number of observations is fixed, but the experiment length is random. The Type II censoring just described is also called Type II censoring on the right. There may be situations where the experiment is not monitored from the very beginning (e.g. times to failure are large) so the smallest observations would be unknown. This subset of Type II censoring is called Type II censoring on the left [5: 46-47] [43: 251-252]. Whether complete or censored samples are used and the type of censoring applied are important factors that affect the goodness-of-fit methodology employed.

2.3 Parameter Estimation

Rarely does a statistician have complete knowledge of the population from which he is sampling; likewise, in goodness-of-fit testing, the parameters of the hypothesized distribution are usually unknown and must be estimated from the sample data. In fact, parameter estimation is one of the

foundational steps in many goodness-of-fit techniques as will be seen. An abundance of estimation techniques have been developed for the Weibull parameters, a few of which will be examined here.

2.3.1 Maximum Likelihood Estimation (MLE). One of the most common and intuitively appealing methods of estimation is Maximum Likelihood Estimation (MLE). This approach finds estimates that maximize the probability of observing the empirical sample data. More formally, it chooses estimates of the parameters that maximize the likelihood function of the observed sample. The likelihood function is simply the joint density function evaluated at the values of the given sample [71: 399].

If $x_1, x_2, ..., x_n$ is a set of independent ordered random samples from a distribution with PDF f(x;$\boldsymbol{\theta}$), where $\boldsymbol{\theta}$ is the vector of parameters for the density function, then the likelihood function is given by

$$L(x_1, x_2, ..., x_n \mid \boldsymbol{\theta}) = \prod_{i=1}^{n} f(x_i \mid \boldsymbol{\theta})$$

For the three-parameter Weibull distribution, the likelihood function is

$$\begin{aligned} L(x_1, x_2, ..., x_n \mid \beta, \delta, \theta) &= \prod_{i=1}^{n} f(x_i \mid \beta, \delta, \theta) \\ &= \left(\frac{\beta}{\theta^\beta}\right)^n \prod_{i=1}^{n} (x_i - \delta)^{\beta-1} \exp\left[-\sum_{i=1}^{n}\left(\frac{x_i - \delta}{\theta}\right)^\beta\right]. \end{aligned}$$

Finding the maximum is simplified by maximizing the log-likelihood function – the natural log of L – which shares the same optimal point as L and has this form:

$$\log(L) = n \log \beta - \beta n \log \theta + (\beta - 1) \sum_{i=1}^{n} \log(x_i - \delta) - \sum_{i=1}^{n} \left(\frac{x_i - \delta}{\theta}\right)^\beta .$$

Taking all the partial derivatives and setting them equal to 0 yields,

$$\begin{aligned}
\frac{\partial \log(L)}{\partial \delta} &= \beta\theta^{-\beta}\sum_{i=1}^{n}(x_i-\delta)^{\beta-1} - (\beta-1)\sum_{i=1}^{n}(x_i-\delta)^{-1} = 0 && (2.1)\\
\frac{\partial \log(L)}{\partial \beta} &= \frac{n}{\beta} - n\log\theta + \sum_{i=1}^{n}\log(x_i-\delta) - \theta^{-\beta}\sum_{i=1}^{n}(x_i-\delta)^{\beta}\log\left(\frac{x_i-\delta}{\theta}\right) = 0 && (2.2)\\
\frac{\partial \log(L)}{\partial \theta} &= -\frac{\beta n}{\theta} + \beta\theta^{-\beta-1}\sum_{i=1}^{n}(x_i-\delta)^{\beta} = 0 && (2.3)
\end{aligned}$$

Solving (2.1), (2.2), and (2.3) simultaneously for β, θ and δ yields the MLEs for the respective parameters.

MLE's have been historically attractive for their excellent statistical properties, summarized below [44: 370] [5: 75-79] [71: 402]:

1. For most distributions, MLEs are unique; there are no alternative values that maximize the likelihood function.
2. MLEs may be biased for small sample sizes, but they are generally asymptotically (for large sample sizes) unbiased.
3. MLEs are asymptotically normally distributed.
4. MLEs are invariant: if $\widehat{\theta}$ is an MLE of θ, and there is some ϕ = h(θ) for a function h, then the MLE of ϕ is h($\widehat{\theta}$). This is one of the most useful properties of MLEs and is elaborated by Casella and Berger's text [15: 293-294].
5. MLEs are strongly consistent, meaning $\lim_{n\to\infty} \hat{\theta} = \theta$.
6. If a sufficient statistic T exists for the estimator of a parameter θ, the MLE will always be some function of T.

Bain discusses several other desirable small- and large-sample properties of MLEs that make them both useful and advantageous [5: 75-79]. Furthermore, MLEs possess good inferential qualities. Antle and Bain found particular functions of the MLEs for location and scale parameters that

have distributions independent of the actual parameters, making them ideal pivotal quantities for inferences on those parameters using MLEs [3].

The high degree of non-linearity in equations (2.1), (2.2) and (2.3) for the Weibull MLEs, however, leads to tremendous computational complexity that has caused many to revert to the simpler two-parameter Weibull, or opt for an alternative to the Weibull model [81: 54,61]. Some, in fact, have argued that the parameter estimates are so elusive that they have rendered the parameters themselves void of meaning [28: 202]. Nonetheless, a variety of studies have addressed the three-parameter problem with diverse numerical techniques. Zanakis and Kaparisis, and Cheng and Amin both provide a good review of the difficulties and various numerical approaches that have been developed [81] [17]. Harter and Moore developed an iterative procedure for complete and type II singly censored samples [35]. Their approach was modified by Gallagher, resulting in a faster algorithm [27]. Wingo utilized a different non-linear optimization technique called modifed quasilinearization and developed an alternate procedure [74]. Lemon added to the field by presenting an estimation procedure for left- and right-censored samples [45]. Later, Zanakis applied a sine-square transformation and compared it to several existing techniques, including Harter and Moore. He found that Harter and Moore's algorithm was more accurate but slower [80]. More recently, Archer combined Harter and Moore's approach with Wingo's with some modifications that converged faster than Harter and Moore, but slower than Wingo [4]. Cohen and Whitten provide a FORTRAN subroutine for the three-parameter MLEs in their text [19: 345-47].

In addition to the computational complexity of the MLE procedures for the three-parameter Weibull, there are other problems with regard to their regularity, existence, and loss of typical MLE properties for particular combinations of parameter values. For instance, when $\beta < 1$, the likelihood function increases without bound as the location parameter approaches the first order statistic of the sample, $\delta \rightarrow x_{(1)}$. Thus, the MLE for δ is $x_{(1)}$ here, but when this happens, the MLEs for β and θ do not exist [19: 27]. Additionally when β is close to 1 but greater than it,

computational problems arise because it is possible that a specific sample leads to a likelihood function with no maximum, and this probability is high when β is close to 1 [17: 395]. When $1 < \beta < 2$ and MLE's do exist, their asymptotic variance-covariance matrix is meaningless, leading to inference problems [38: 310]. Finally, the usual asymptotic properties of MLEs do not hold for the three-parameter Weibull unless $\beta > 2$. Consequently, Cohen and Whitten recommend against using MLEs for Weibull parameters unless $\beta > 2.2$ [19: 27].

In the case of known shape parameter, the problem becomes slightly less troublesome. Rockette et al. proved that with shape known, the MLEs of location and scale do exist and are unique if $\beta \geq 1$ [56]. With iterative procedures, like those mentioned above, one simply omits the cyclic estimation of the (known) shape parameter, substituting the true value [35: 641]. Another approach would be to substitute the known shape parameter into equations (2.1) and (2.3), reducing the problem to solving only two non-linear equations simultaneously, as Bush describes [12: 18]. The assumption of known shape may seem unrealistic, but often it is possible to have *a priori* knowledge of this parameter from theoretical or empirical grounds. Harter and Moore, for example have tabled more than a dozen applications in which the shape parameter for the Weibull model is known [37: 101]. In other cases, a statistical test assuming known shape can actually be used to test the shape assumption.

2.3.2 Other Estimation Methods. In response to the problematic aspects of Weibull MLEs, a host of alternative estimation methods have been proposed. Engelhardt and Bain introduced good linear unbiased estimators (GLUEs) for the extreme value distribution (a log-transformed two-parameter Weibull) which were simple to compute and nearly equivalent to MLE's in a heavy censoring situation, but do not have minimum mean square error (MSE), as MLEs do [25]. They later developed modifications to these GLUEs, which did minimize the MSE but at the expense of unbiasedness. These MGLUEs, as they are called, appeared to agree closely with the MLEs [5: 257].

However, both types of estimators do not exist for the three-parameter case unless the shape or location parameter are known. Their text provides formulas for both such cases [5: 284-85].

A different iterative approach, which, like the MLE techniques solves three non-linear equations in three unknowns, but is well-defined in cases where ML estimation fails, was published by Cheng and Amin. This technique, called maximum product of spacings (MPS) estimation, retains the useful properties of ML estimation (e.g. consistency, asymptotic normality) but does not suffer from the aforementioned pitfalls of MLEs. They are, however, as computationally expensive as the MLEs [17].

Smith and Naylor compared MLEs with a Bayesian estimation approach using two sample data sets, and concluded that the Bayesian method produced better estimators than the MLEs [65]. Sinha and Sloan showed that these estimators performed best on large sample sizes, where they had smaller variances than MLEs [64].

Moment estimators (MEs), found by equating the first three sample moments to corresponding population moments, are given by Cohen and Whitten. They require less computational effort than the MLEs and do not face the numerical difficulties when $\beta < 1$, but have larger variances than MLEs [19: 31-32]. To address this limitation, they generated modified moment estimators (MMEs) in which the third moment equation for the MEs is replaced with one in which the sample first order statistic is equated to its expectation. These MMEs do not encounter difficulties with particular parameter values like the MLEs and are unbiased, but they do not perform as well as MLEs when skewness is near 0. FORTRAN code for MEs and MMEs is provided in their text [19: 32,341-44].

Another technique more recently applied to the Weibull case is miminum distance estimation (MDE). This methodology minimizes the distance between the cumulative distribution function (CDF) values of sample data and the empirical distribution function (EDF). The CDF values are computed from the hypothesized distribution the data belongs to (e.g. the Weibull). The distance measure is quantified with an EDF test statistic (described below), and the parameter

being estimated is iteratively adjusted until the test statistic value is minimized [20: 2-1 to 2-4]. Gallagher and Moore argue that in the case of the three-parameter Weibull, combining the MD estimate of the location parameter with ML estimation of shape and scale is preferable to using MLEs for all three [27: 575-580]. Other estimation methods that are beyond the scope of this project include Wyckoff, Bain and Engelhardt's estimators (WBE), Zanakis estimators (ZNKS) and extended maximum likelihood estimators [19: 44-45] [38: 91].

In spite of this rich collection of viable alternative estimation methods, most goodness-of-fit methodologies use the MLEs as the basis for the tests. Shortcomings aside, the desired statistical properties of MLEs, their intuitive appeal, the wealth of experience with them, and their strong performance in various tests, makes them popular. With the exception of Wozniak and Li's use of the GLUEs and MGLUEs in their Weibull goodness-of-fit studies [78] [77], and Crown's and Yucel's application of the MDEs, few other published investigations have put these alternative methods to use. In fact, Wozniak found that tests using MLEs were more powerful than those using GLUEs or MGLUEs in the case of complete samples (though not so for censored samples) [77: 160]. Cheng and Amin note that when MLEs do exist, they tend to outperform alternate methods [17: 395]. Lending further support to their use, Winkler comments that, "the principle of maximum likelihood is in the spirit of empirical science, and it runs throughout the methods of statistical inference" [75: 350]. Hence, the most common and popular goodness-of-fit approaches rely on MLE parameter estimation and consequently require a substantial degree of complexity to evaluate and may encounter numerical difficulties. The sequential approach investigated in this research is significant, then, in that it bypasses these parameter estimation headaches by relying only on the sample moments of the observed data.

2.4 Skewness and Kurtosis

Two characteristics of distributions that are often employed to describe the shape of the PDF are the skewness and kurtosis. A distribution in which more of the probability mass lies to one side of the midrange than the other and one tail is resultingly longer than the other, is considered skewed. The degree to which it is skewed is quantified by the skewness value. Kurtosis, on the other hand, is normally considered to be a measure of "peakedness", curvature, or tail weight of the PDF. Distributions with more mass concentrated at a central location and in longer tails have high kurtosis values, while flatter PDFs with less mass in short tails reflect lower kurtosis.

Typical measures for skewness and kurtosis of a population are given by the third and fourth moments about the mean, expressed by $E(x-\mu)^3$ and $E(x-\mu)^4$ respectively, where E is the expected value operator. To render these quantities invariant to changes in location and scale, they are divided by σ^3 and σ^4 repsectively, and thus standardized [60: 102]. The resulting measures, the third and fourth standardized moments, then are given by [21: 317]

$$
\begin{aligned}
\sqrt{\beta_1} &= \frac{E(x-\mu)^3}{\sigma^3}, \text{ (Skewness) and} \\
\beta_2 &= \frac{E(x-\mu)^4}{\sigma^4} \text{ (Kurtosis).}
\end{aligned}
$$

Frequently, kurtosis is "corrected" by subtracting 3 from it (β_2-3) so that it becomes a comparative measure relative to the normal curve. For normally distributed populations $\beta = 3$; hence, the corrected measure is 0. With this convention, kurtosis becomes a measure of departure from "normal" kurtosis, with negative values associated with flatter than normal curves, and positive values indicate curves more peaked than a normal PDF [18: 20]. For this paper, the "uncorrected" kurtosis measure β_2 will be utilized.

For symmetric distributions, $\sqrt{\beta_1} = 0$, while a positive skewness value ($\sqrt{\beta_1} > 0$) corresponds to skewness to the right, meaning a longer tail on the right side of the distribution. A

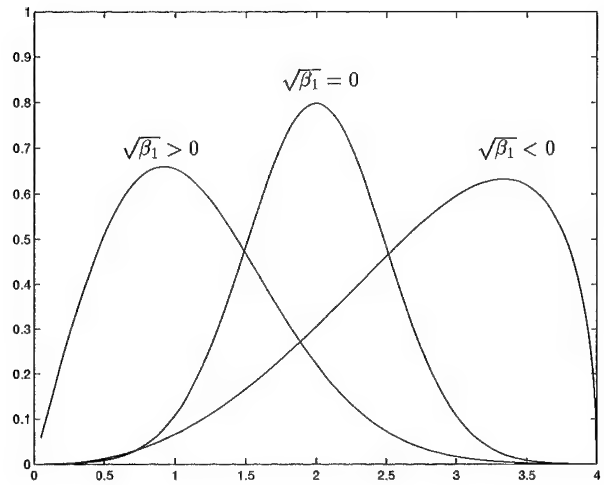


Figure 2.4 PDFs with Differing Skewness

negative skewness ($\sqrt{\beta_1} < 0$) on the other hand, indicates skewness to the left, or a longer tail on the left side of the curve. The differences are highlighted with the curves in Figure 2.4.

Similarly for kurtosis, the reference is a normal population, which has a $\beta_2 = 3$ and is considered mesokurtic. If $\beta_2 > 3$, the distribution tends to have a higher peak and heavier tails and is called leptokurtic, while those with $\beta_2 < 3$ have broader peaks and lighter tails and are referred to as platykurtic. Examples of curves with different kurtosis values are given in Figure 2.5 [21: 317]. Note that the platykurtic example in the figure exhibits more peakedness than the normal distribution but has no tails to speak of, demonstrating that kurtosis is heavily influenced by tail thickness.

Given a random sample of size n, $X_1, X_2, ..., X_n$, from a given distribution, one may evaluate the population characteristics in terms of sample moments. The first sample moment, the sample mean, is given by

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i .$$

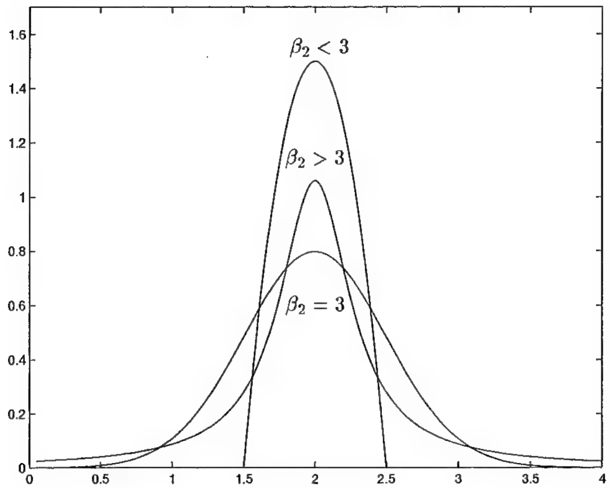


Figure 2.5 PDFs with Differing Kurtosis

The remaining sample moments about the mean are defined as follows:

$$m_k = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^k, \qquad k = 2, 3, 4, ...$$

With these expressions, the sample skewness ($\sqrt{b_1}$) and sample kurtosis (b_2) are defined by

$$\begin{aligned} \sqrt{b_1} &= \frac{m_3}{m_2^{3/2}}, \quad \text{and} \\ b_2 &= \frac{m_4}{m_2^2}. \end{aligned}$$

These measures are invariant to location and scale changes [21: 317] [69: 279-80].

A great deal of effort has gone into characterizing the distributions of $\sqrt{b_1}$ and b_2 for samples from normal populations, and these moments have been widely developed as goodness-of-fit test statistics for normality testing [69: 375-97]. No research, however, has investigated their use in goodness-of-fit testing for Weibull samples. The purpose of this paper is to address that void.

Of note as well is the significant amount of work in the literature committed to interpreting the kurtosis statistic, which has prompted some debate. Typically the emphasis in describing

kurtosis is that it characterizes peakedness of a distribution. Chissom, however, demonstrated that kurtosis is clearly dependent on both the peak of the distribution and its tails, with the tails bearing substantial influence on the value of the fourth moment. He notes that in cases with truncation in the tails, a very peaked distribution may have low kurtosis, even negative "corrected" kurtosis. In other words, the "tailing-off effect" must be present in the tails to give rise to high kurtosis . From this, he argues that a leptokurtic distribution does not necessarily imply that $\beta_2 > 3$ [18].

Balanda and MacGillivray conducted an extensive review of the concept of kurtosis and its interpretation. They concluded that the shape characteristic called kurtosis is vague, but can best be defined by "the location- and scale-free movement of probability mass from the shoulders of a distribution to its center and tails" [7: 111]. Due to the averaging nature of moments, they remark that the relationship between β_2 and distributional shape is blurred; indeed, it is not always a good indicator of shape, evidenced by the fact that a given value of β_2 can correspond to various distributional shapes. Peakedness and tail weight, then, are better seen as components of kurtosis [7]. Later, the same authors formalized their arguments and definitions with an extensive discussion of theory and applications regarding kurtosis [8].

Some of the vagueness noted above with kurtosis also plagues skewness to some degree. Royston evinced that both the conventional measures for sample skewness and kurtosis ($\sqrt{b_1}$ and b_2) have substantial theoretical and practical drawbacks in assessing distribution shape. Their interpretations are not always clear, they are sensitive to small changes in the tails, and are susceptible to outliers. Consequently, he proposes two alternative statistics t_3 and t_4 to replace the convential measures in routine use [58].

In spite of these very valid concerns, the $\sqrt{b_1}$ and b_2 statistics have been used extensively in the goodness-of-fit field and have proven useful [21: 316]. Identifying their utility in assessing Weibull model adequacy will be the focus of this effort, and the drawbacks mentioned will certainly warrant further investigation if their performance here is not up to par.

2.5 Goodness-of-Fit-Tests

Goodness-of-fit tests are formalized statistical tests that measure how well a given sample of data agrees with an hypothesized distribution for its parent population. While graphical methods do exist, most established tests utilize particular test statistics. For such tests, the null hypothesis (H_0) is that the given sample is from a particular stated distribution, and in most cases the alternative (H_a) states that H_0 is false, but may actually specify an alternate distribution. Stephens distinguishes between cases in which the null hypothesis is completely specified, called a simple hypothesis, and a composite hypothesis, where none or only some of the parameters are specified. In most cases, H_a is composite. One aspect of goodness-of-fit tests that distinguishes them from conventional hypothesis tests is the fact that the objective is usually to accept H_0 (or fail to reject) as opposed to rejecting it. The problem can be summarized as follows: [69: 1-2]

Given a random sample $X_1, X_2, ..., X_n$, determine if H_0: The X_is are IID random variables with distribution function F ($X \sim F(x)$) versus H_a: $X \not\sim$ F(x)

The general procedure can be summarized in the following steps:

- formulate an hypothesized distribution
- specify or estimate the parameters for the distribution
- calculate the test statistic
- evaluate acceptance / rejection of H_0

Goodness-of-fit tests can be further segregated into completely specified and modified tests. Completely specified tests are those in which all the parameters in H_0 are known, whereas in modified tests, the parameters must be estimated from the sample data. The difference is significant – critical values for the same test statistics of certain tests are different depending on whether or not parameters must be estimated. Using the critical value tables for a completely specified case

when parameters are actually estimated will bias the test toward acceptance to the point that it renders the test almost useless [76: 115].

Numerous goodness-of-fit tests and test statistics have been developed in the last century, all with various strengths and weaknesses. D'Agostino and Stephens' text provides a comprehensive review with extensive bibliographic references [69]. A brief review of some of the key techniques follows.

2.5.1 Chi-Square Test. One of the classical and most widely applicable tests presented in almost every statistical textbook is the Chi-Square (χ^2) Test. Developed by Pearson in 1900, it formalizes the intuitive approach of comparing the histogram of the sample data to the shape of the hypothesized distribution PDF. The test is useful for both continuous and discrete data and basically entails grouping the data into a collection of cells or intervals and then measuring the deviations between the observed frequencies in the cells and the expected frequencies determined by the hypothesized distribution. If the differences, compiled with the χ_0^2 test statistic, are large, the null hypothesis is rejected. For a formal description see Banks, et al. [9: 375-76] or Law and Kelton [44: 382-86].

Grouping the sample data into a set of somewhat arbitrary cells, however, discards some information; hence, these tests are usually less powerful than other types [69: 63] [67: 730]. Furthermore, they are not recommended for sample sizes less than roughly 20 [9: 377], so in situations with limited data, other tests must be used. An additional drawback is that the choice of cell numbers will vary the outcomes of the test, meaning the test is not unique [76: 113]. Thus, in spite of their wide applicability, Chi-square tests are usually employed for discrete distributions and are not used for continuous goodness-of-fit problems.

2.5.2 EDF Tests. A much more pervasive and extensive category of tests are the EDF (Empirical Distribution Function) Tests. These tests are based on a comparison of the hypothesized

CDF $F(x)$ to the empirical distribution function $F_n(x)$ derived from the sample data. EDF test statistics basically quantify the difference between the two for a measure of agreement.

The EDF itself is a step function calculated from the sample data that approximates the population distribution function. An illustration is provided in Figure 2.6. Given a random sample of size n, $X_1, X_2, \ldots, X_n$, the EDF, $F_n(x)$ is defined as

$$F_n(x) = \frac{\text{number of sample observations } \leq x}{n} \qquad -\infty < x < \infty$$

If we let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the corresponding order statistics, then as x increases, the function $F_n(x)$ will increase by $\frac{1}{n}$ at each point x equals an order statistic. So, we may more precisely define the EDF as [69: 97,99]

$$F_n(x) = \begin{cases} 0 & \text{if} \quad x < X_{(1)} \\ \frac{i}{n} & \text{if} \quad X_{(i)} \leq x < X_{(i+1)} \qquad i = 1, \ldots, n\text{-}1 \\ 1 & \text{if} \quad X_{(n)} \leq x \end{cases}$$

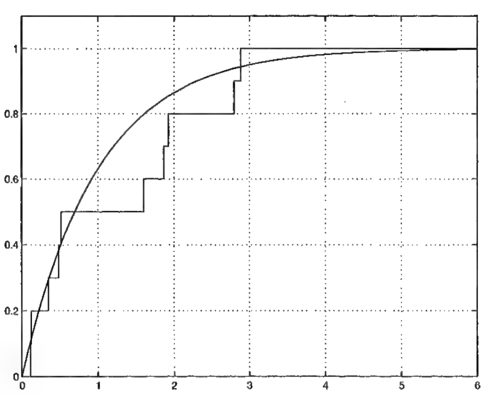


Figure 2.6 EDF for a Sample of Size 10 from an Exponential Distribution

Two basic classes of EDF statistics exist: the supremum class and the quadratic class. The supremum class includes statistics based on the largest 'vertical' differences between $F_n(x)$ and

$F(x)$ in the two cases where $F_n(x)$ is greater than $F(x)$ or less than $F(x)$. Hence, we have

$$D^+ = \max_x \left\{ \tfrac{i}{n} - F(x) \right\}, \text{ and}$$
$$D^- = \max_x \left\{ F(x) - \tfrac{i-1}{n} \right\}.$$

The most well-known supremum EDF statistic is the Kolmogorov-Smirnov D, given by,

$$D = \max\left\{D^+, D^-\right\} = \sup\left|F_n(x) - F(x)\right|.$$

Another popular supremum statistic is Kupier's V, where

$$V = D^+ + D^-.$$

The quadratic class of EDF statistics are based on integrating the squared differences between $F_n(x)$ and $F(x)$ over all x. They belong to the Cramér-von Mises family given by

$$Q = n \int_{-\infty}^{\infty} \left(F_n(x) - F(x)\right)^2 \psi(x) dF(x)..$$

The function $\psi(x)$ is a weighting function on the squared differences. When $\psi(x) = 1$, the result is the Cramér-von Mises statistic W^2, and when

$$\psi(x) = \frac{1}{F(x)\left(1 - F(x)\right)},$$

the result is the Anderson-Darling statistic, A^2. This particular choice of the weighting factor emphasizes the discrepancies in the tails of the distribution.

Computational formulas for these statistics are found by applying the Probability Integral Transformation (PIT) on the ordered sample data. First, the data must be sorted in ascending order; next, the PIT yields

$$Z_{(i)} = F(x_{(i)}) \qquad i = 1, 2, ..., n$$

where F(x) is the hypothesized CDF. This will result in an ordered set of Z values, $Z_{(1)} \leq Z_{(2)} \leq ... \leq Z_{(n)}$. Using these Z values, the computational formulas for W^2 and A^2 are given by D'Agostino and Stephens as [69: 100-01] [67: 731]

$$\begin{aligned} W^2 &= \frac{1}{12n} + \sum_{i=1}^{n} \left(Z_i - \frac{(2i-1)}{2n} \right)^2, \qquad \text{and} \\ A^2 &= -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left[\log Z_i + \log\left(1 - Z_{n+1-i}\right) \right]. \end{aligned}$$

In cases where the hypothesized distribution is completely specified, the EDF statistics are distribution free; in other words, they do not depend on the hypothesized distribution. Consequently, a single table of critical values may be used for all distributions for a given test statistic [76: 115]. However, when some or all the parameters must be estimated, which is usually the case, the distribution of the EDF statistics depends on the hypothesized distribution, the sample size, the values of the estimated parameters, and the method of estimation [69: 102]. The result is that one needs separate tables of critical values for each condition noted.

This fact prevented the widespread use of EDF statistics initially [67: 730]. David and Johnson, however, demonstrated that in cases where only the location and scale parameters were unknown and were estimated with invariant estimators, test statistics that depend on the PIT do not depend on the true values of location and scale, only on the functional form of the CDF [24]. This finding simplified the use of EDF statistics where only location and scale parameters had to be estimated since it established that they were location and scale invariant when the proper estimators were employed. If this is the case, the distribution of the EDF statistics depends only

on the sample size, the estimation method, and the family of the distribution specified in H_0. Hence, critical values can be tabled for a given distributional family and sample size based on the estimation approach taken. The levels will apply for all values of location and scale. Stephens has provided tables for several EDF statistics for completely specified cases and for the Normal and Exponential distributions with estimated parameters [67: 732-33]. Green and Hegazy built tables for Uniform, Normal, Laplace, Exponential and Cauchy distributions when parameters were estimated with several invariant approaches [29: 207]. Littell, et al. published tables for the A^2, W^2, D, and other non EDF statistics for the two-parameter Weibull using maximum likelihood estimators (MLEs) [46: 257-69].

Numerous studies have generated critical values for several EDF statistics for the two- and three-parameter Weibull distribution. In two-parameter cases where the shape parameter must be estimated, the raw data must first be log transformed before the goodness-of-fit test is conducted. This transformation converts the Weibull shape and scale parameters into scale and location of the extreme-value distribution, which permits application of David and Johnson's findings. If the data were indeed Weibull, the transformation converts it to extreme-value data so that the test for the extreme-value distribution becomes equivalent to a Weibull goodness-of-fit test [78: 133-34]. Stephens calculated asymptotic critical values for W^2 and A^2 for the extreme value distribution in the completely specified case and when parameter estimates are found by maximum likelihood (ML) [68]. For finite sample sizes, Littell et al. generated critical values for modified EDF tests on the two-parameter Weibull using D, A^2, W^2 , and some non-EDF statistics when parameters are replaced by MLEs [46]. Chandra et al. found percentage points for the Kolmogorov-Smirnov statistics ($D+$, $D-$, and D) as well as Kupier's V using ML estimates of Weibull parameters [16]. In the three-parameter case with a known shape and MLEs for location and scale, Bush addressed modified EDF tests with W^2 and A^2 [12]. While this earlier work focused on complete samples, Aho, Bain and Engelhardt tabled critical values for D, V, W^2, and A^2, for the two-parameter Weibull with ML parameter estimates for various levels of censoring in the sample data [1] [2].

Later, Wozniak contributed by dealing with situations in which other forms of estimation were used instead of MLE. She utilized Engelhardt and Bain's good linear unbiased estimators (GLUEs) and their modified versions (MGLUEs) for the two-parameter case with complete and censored samples and built critical value tables for several EDF statistics for each estimation approach [78]. Crown tabled A^2 and W^2 values for the three-parameter Weibull with known shape using Minimum Distance Estimation (MDE) of parameters [20]. And Yucel modified the A^2 and W^2 tests for the three-parameter case when all three parameters were unknown and estimated with a combination of ML and MD estimation methods [79]. From this brief sampling, it is obvious that the literature is replete with EDF tests applied to the Weibull distribution, which is indicative of the popularity of these test statistics for goodness-of-fit.

Power studies comparing the performance of these EDF test statistics have given pronounced support to the superiority of the Cramér-von Mises W^2 and the Anderson-Darling A^2 tests over other tests. Performance comparisons among tests can be challenging to interpret because powers often vary based on sample sizes and the types of alternate distributions considered. For a given null hypothesis and identical alternatives, one test may outperform another for one sample size, while the roles are reversed for a different sample size. Nevertheless, general observations can still be useful. Green and Hegazy provided a table recommending which EDF test statistics to use when testing a particular distribution against several alternatives. Although they did not consider the Weibull specifically, they highly favored the W^2 and A^2 - based tests for testing the Uniform, Normal, Laplace, Exponential and Cauchy distributions [29: 207]. Woodruff and Moore documented the superior power of the Cramér-von Mises and Anderson-Darling over other EDF tests for a wide range of distributions, including the Weibull, Exponential, Gamma, Logistic, and Laplace. Between W^2 and A^2, preference was a function of which distribution was being tested; for the Weibull, the Anderson-Darling tended to outperform W^2 [76: 116-18]. Specifically for the two-parameter Weibull with ML parameter estimates, Littell et al. examined the power of the D, A^2, and W^2 EDF tests and two additional tests; the S statistic developed my Mann,

Scheuer, and Fertig (MSF) and a T statistic introduced by Smith and Bain. They found that while the MSF test did well, the A^2 and W^2 were advisable when one had no prior knowledge of alternatives [46: 257-69]. Aho, et al. investigated the power of the K-S, Kupier's V, and W^2 EDF tests with the MSF test for the two-parameter Weibull using MLEs in a censored sampling environment. Again, they found the Cramér-von Mises test preferable to the other EDF tests, and the MSF test a close competitor [1: 68]. For heavily censored samples, they found similar results in their later study [2: 223]. Wozniak's power study on the two-parameter Weibull using three different parameter estimation approaches, compared the D, V, W^2, and A^2 tests and illustrated the fact that power was a function of estimation method. She still found that the W^2 and A^2 displayed consistently better power than the others [77: 153-61]. For the three-parameter Weibull case with a known shape parameter and ML estimation, Bush, et al. compared the Chi-square, K-S, W^2, and A^2 tests. Consistent with the other studies noted, they found the Cramér-von Mises and Anderson-Darling tests the best, with the W^2 being most powerful for the $\beta = 1$ case, and the A^2 superior for $\beta = 3.5$ [13]. Given this wealth of evidence, this paper will utilize the Cramér-von Mises and Anderson-Darling tests and the power results documented by Bush and Wozniak as the benchmarks to compare with the power of the sequential test developed herein.

2.5.3 Moment Tests. The standardized third and fourth sample moments $\sqrt{b_1}$ and b_2, which, like EDF statistics, are invariant to location and scale of a given distribution, have also been employed widely for goodness-of-fit applications. They offer the advantage of not requiring parameter estimation. Their use however, has been almost exclusively limited to tests for normality or departures from it. D'Agostino and Stephens provide a thorough overview of the history of these tests [69: 279-318; 375-91]. For all normal distributions, the population moments $\sqrt{\beta_1} = 0$, and $\beta_2 = 3$. Since these values are known, samples that are indeed normal should reflect these values in their sample skewness and sample kurtosis measures. For small sample sizes in particular, though, the distribution of the sample moments $\sqrt{b_1}$ and b_2 may vary widely from the true population

values. But extensive studies have adequately approximated these distributions, facilitating their use as test statistics for normality [54: 233] [69: 280].

It is often possible to specify the way data may depart from the distribution specified by the null hypothesis, and thus qualify the types of alternatives to consider testing against. Such tests are called directional tests, as opposed to omnibus tests, which are sensitive to any departure from H_0 for cases in which one has no prior knowledge of the alternatives [54: 232]. Intuitively, $\sqrt{b_1}$ would be useful in a directional test for normality against skewed alternatives, and b_2 could be appropriate for such tests against departures in kurtosis. Non-normal samples from skewed or asymmetric distributions will have $\sqrt{b_1}$ values different from 0, but may have similar weights in the tails to the normal distribution (b_2 values near 3). On the other hand, non-normal samples from symmetric unimodal distributions ($\sqrt{\beta_1} = 0$) can be characterized by non-normal kurtosis ($\beta_2 \neq 3$). Hence $\sqrt{b_1}$ and b_2 should also be able to be usefully employed in concert with one another as test statistics to discriminate between normal and a variety of non-normal populations [69: 280].

In two comparative studies, Shapiro and Wilk and Shapiro et al. used $\sqrt{b_1}$ and b_2 in separate two-sided directional tests for normality to contrast their own omnibus W test [61] [62]. They found that $\sqrt{b_1}$ was sensitive in detecting highly skewed and long tailed distributions, but was not as effective against symmetric or asymmetric finite range distributions. On the other hand, b_2 demonstrated good power against symmetric, long tailed distributions and against finite range cases, but stumbled on skewed samples [62: 1366-67]. They note that b_2 would be useful to augment $\sqrt{b_1}$ in tests for normality [61: 608]. Their proposed W statistic fared better than $\sqrt{b_1}$ and b_2 separately, but they concluded that utilizing both together was a worthy competitor [62: 1371]. Snedocor and Cochran also address these directional tests in their text [66: 78-81] and Sachs has compiled tables of critical values for the $\sqrt{b_1}$ and b_2 statistics for sample sizes ranging from n = 7 to 2000 [60: 326]. D'Agostino and Stephens conclude that the $\sqrt{b_1}$ and b_2 tests have excellent sensitivity over a wide range of alternatives [69: 404]. Additionally, D'Agostino and Pearson derived cumulative

probability curves for b_2 for sample sizes n = 20 (5) 50, 100, 200 and suggested two omnibus tests for normality based on $\sqrt{b_1}$ and b_2. One of these statistics is given by $K^2 = X^2(\sqrt{b_1}) + X^2(b_2)$ where $X^2(\sqrt{b_1})$ and $X^2(b_2)$ are standardized normal equivalent deviates and K^2 is approximately χ^2 with two degrees of freedom [22] [23]. Bowman and Shenton modified this statistic using a different approximation to $X(b_2)$ and generated contour plots in the ($\sqrt{b_1}$, b_2) plane for the test with numerous sample sizes and significance levels to facilitate its application [10]. D'Agostino and Stephens' text provides an assortment of similar contour plots in the ($\sqrt{b_1}$, b_2) plane for the K^2 and other similar test statistics.

Pearson et al. discuss another simple omnibus test called the R-test, which is essentially a sequential test using the separate $\sqrt{b_1}$ and b_2 tests one after another. Normality is rejected if either test leads to rejection. If the tests are conducted at the same significance level α', they found that a good approximation to the overall significance level is given by

$$\begin{aligned} \alpha &= 4\left\{\alpha' - (\alpha')^2\right\}, \qquad \text{and thus} \\ \alpha' &= \frac{1}{2}\left\{1 - (1-\alpha)^{1/2}\right\}. \end{aligned} \tag{2.4}$$

which would hold if $\sqrt{b_1}$ and b_2 were independent. Although they are uncorrelated, they are not independent, so the test is actually more conservative than the overall α-level found above. Their power study indicated that the R-test fared well, but not better than some other tests for normality. The name R-test derives from the fact that the test can be viewed as using a rectangle in the ($\sqrt{b_1}$, b_2) plane, defined by the upper and lower-tail critical points of $\sqrt{b_1}$ and b_2 as the criteria for rejection of normality [54].

These moments were also adapted in Oja's presentation of a series of tests for normality based on particular orderings of skewness and kurtosis [52]. Mardia extended the application of skewness and kurtosis tests from univariate to multivariate normality tests [48]. D'Agostino and Stephens review the power results of most of the above-mentioned tests in their text [69: 404].

Although there are a plethora of moment-based tests for normality, they have not been used extensively in spite of their excellent utility. D'Agostino et al., noting this failure, recommend the tests based on $\sqrt{b_1}$ and b_2 as powerful and informative goodness-of-fit tools [21]. Hopkins et al. likewise argue that skewness and kurtosis have tremendous utility in normality tests, but are largely ignored [41]. Horswell and Looney agree that skewness and kurtosis together can be a powerful test for normality against widely ranging alternatives, but challenge D'Agostino's claims regarding the diagnostic properties of skewness, noting that it is very sensitive to kurtosis; in other words, certain values of kurtosis (b_2) may weaken the power of the $\sqrt{b_1}$ test statistic [42].

The abundance of powerful skewness and kurtosis-based tests for normality would suggest that similar studies would have produced parallel tests for other distributions, including the Weibull. This, however, is not the case. Hahn and Shapiro, noted by Rousu, demonstrated that the Weibull skewness and kurtosis are functions of the shape parameter and are independent of location and scale [57]. This would seem to indicate that in cases where shape was known, the Weibull PDF would share the characteristic of fixed skewness and kurtosis values that made moment tests viable for the normal PDF. Although he suggests a crude adequacy of fit test using a region in the (β_1, β_2) plane to compare with sample $\sqrt{b_1}$ and b_2 values, Rousu did not expand on the concept [57].

More support for the utility of skewness and kurtosis values in discriminating among distributions comes from the field of adaptive robust parameter estimation. This field of study is concerned with methods of estimating parameters of a sample that are relatively insensitive to assumptions concerning the nature of its underlying distribution. An example would be utilizing some goodness-of-fit-type test statistic to discriminate among a specified set of distributional families, with particular values of the statistic indicating which family is most appropriate for the data. Then one uses the MLEs for the particular family identified as the parameter estimates. A particularly useful discriminatory statistic has proven to be the sample kurtosis (b_2). Hogg demonstrated its effectiveness in discriminating between symmetric, unimodal distributions, specifically the Normal,

Uniform, Logistic, and Double Exponential [40]. He later proposed another statistic, Q, as a more powerful discriminant than kurtosis [39], and Harter developed robust estimators for location and scale of symmetric, unimodal distributions using both b_2 and Q as discriminators [34]. Rugg found that while both b_2 and Q were effective, Q performed better. He also attempted to include sample skewness to increase the power of b_2 and Q, but it did not help [59], which is not surprising since he was considering only symmetric, unimodal distributions. In another application, Forth used the kurtosis of residuals in a robust regression procedure to determine which form of the regression coefficients to select [26].

This brief sampling of the arena of robust statistics adds credence to the utility of kurtosis, at least, as a discriminatory tool among distributions, particularly among those with similar skewness characteristics. This observation, combined with the strong evidence from the goodness-of-fit literature on moment-based tests for normality, makes it seem logical that these sample moments could prove useful in determining the adequacy of other distributions as well, namely the Weibull. There are detractors, however. Law and Kelton claim that they did not find kurtosis to be tremendously useful in differentiating between distributions [44: 360]. Nonetheless, the absence of published literature addressing the effectiveness of sample skewness and kurtosis in goodness-of-fit testing for the Weibull in contrast to the positive findings for the normal, suggest a potentially fruitful avenue of investigation. The R-test noted earlier serves as inspiration to develop a similar approach for the Weibull distribution.

2.5.4 Sequential Tests. Sequential goodness-of-fit tests are not "new" tests *per se*; they simply employ existing tests in sequence with the aim of acheiving a more consistently powerful test against a wider range of alternative hypotheses — in other words, an omnibus test. There is a pronounced paucity of literature on sequential tests with few notable exceptions. Pearson's R-test for normality mentioned earlier is one example of a simple application of sequential tests. Onen broke new ground in implementing a series of sequential goodness-of-fit tests for the Cauchy distribution

using various combinations of EDF statistics. He found that against symmetric distributions, the power of a sequential test is always between the power of the two individual tests at identical α-levels; hence, it improved the power of the weaker test in the pair. Against, non-symmetric distributions, however, the sequential tests did not fare as well [53]. More recently, Gunes et al. employed six sequential modified goodness-of-fit tests for the inverse Gaussian distribution using EDF tests and Watson's W test. They concluded that a sequential procedure which combines tests that are powerful at opposite levels of symmetry can be a valuable alternative to a single test [31].

The challenge in developing and using sequential tests is in the determination of attained significance levels for the overall test. If one test in a sequential procedure is conducted at an α_1 level, and the other test at an α_2 level, Bonferroni's inequality informs us that the overall level of the combined test is $\alpha \leq \alpha_1 + \alpha_2$ [69: 390]. In other words, the sequential test has significance level no larger than the sum of the two levels of the tests that constitute it. Bonferroni's inequality provides an upper bound for the overall significance level but does not yield an exact value. For Pearsons' R-test, an overall significance level was approximated (see Equation 2.4), but was contingent upon starting with identical levels with each test and making an independence assumption that did not hold [69: 390]. Determining an exact value for the actual attained significance level of this test and other sequential tests is elusive. In fact, for a given desired overall α-level, there are numerous possible combinations of levels for the individual tests that could yield it. Onen and Gunes used Monte Carlo techniques to develop tables of empirically observed attained significance levels for their sequential tests, allowing the reader to identify several levels in the given tables that would lie close to a desired value and then to choose which combination of α-levels for the individual tests would be best for a particular situation. If the reader had some additional insight into his data that might lead him to presume it would be skewed, for example, he could pick a greater significance level (larger α) for the test that was more powerful against skewed alternatives [53: 5-7] [31: 74]. Another approach to the significance level dilemma is to employ contour plots of overall significance

regions similar to those presented by D'Agostino and Stephens for normality tests in the $(\sqrt{b_1}, b_2)$ plane [69: 282] with the difference being that the axes could represent α-levels for each test.

Another concern in sequential tests is the power of the overall test. Normally, power increases as α increases, since Type II error (β) decreases [60: 126]. So, combining tests may increase power, but not at the same significance levels as the original tests. As Gunes et al. point out, the power of a sequential test usually lies between the powers of the tests that consitute it at the same significance level [31: 73]. The logical question, then, is why use a sequential test if it tends to be less powerful at a given α-level than the best test that it combines? The answer lies in the fact that the sequential test may be more powerful on average against a wider range of alternatives than either of the two separate tests. If one combines a test that is powerful against symmetric alternatives but weak on skewed alternatives with a test that is effective against skewed alternatives, the result may be a useful omnibus test that will yield better results against a broad spectrum of alternatives.

It is this capability that this project seeks to exploit. It has been discussed how sample skewness could discern against distributions that differ from the normal in terms of skew, while kurtosis effectively discriminated among symmetric distributions that varied in degree of tail-weight or peakedness. Combining the two in a sequential test may prove to be a useful and fairly simple omnibus test for goodness-of-fit for the Weibull distribution.

2.6 Conclusion

A wealth of research has been conducted on the widely-used and flexible Weibull distribution, and several powerful goodness-of-fit tests have been developed for it. Due to the complex nature of procedures to estimate Weibull parameters, however, many of these tests are very expensive computationally and are sometimes plagued by numerical difficulties. Moment-based tests avoid these challenges and are well-documented for the normal distribution, demonstrating fairly good power. No such approach has been undertaken for the Weibull, though. A sequential goodness-of-

fit procedure for the Weibull based on the standardized third and fourth sample moments has the potential of providing good power against a range of alternatives without the typical computational outlays.

III. METHODOLOGY

3.1 Introduction

In the development of a goodness-of-fit test that uses a particular test statistic, as most formal tests do, the first order of business is to determine the sampling distribution of the said test statistic for a particular null hypothesis. From this point, critical values at specified significance levels of the test can be derived and tabled for use. These tables formalize the employment of the test. Constructing a sequential testing procedure involving two test statistics as purposed here, however, complicates this process somewhat. Although the critical values for the individual tests may be tabled, the overall attained significance levels for the combined test are not as straight forward. Indeed, as noted earlier, there will be various combinations of significance levels for the individual tests that will yield the same overall significance level in the sequential test. Hence, a significant portion of the effort in developing a sequential test is determining the attained significance levels for a host of combinations of levels for the tests that make it up.

Once the test has been developed in this fashion, it is vital to determine its utility compared to existing tests. The primary discriminant in this comparison is the power of the test, or the probability of correctly rejecting a false null hypothesis. Consequently, a power study should be undertaken to evaluate the new test's power versus a wide range of sample data from alternate hypotheses. Ideally, the new test will have good power against a broad range of alternate hypotheses when compared to existing tests. In reality, most tests' power fluctuates significantly based on the alternates considered, and rarely does one procedure demonstrate superiority over all others in all cases [69: 2].

The objective here is to formally develop a sequential goodness-of-fit test using sample skewness and kurtosis for a range of Weibull sample sizes with known shape, and evaluate its power against various alternative distributions compared to the powers of existing EDF tests; namely the Kolmogorov-Smirnov, the Cramér-von Mises, and the Anderson-Darling procedures.

3.2 Critical Value Determination

3.2.1 Monte Carlo Methods. Typically in goodness-of-fit testing, determining the sampling distribution of test statistics is mathematically intractable or simply too difficult to express in closed form. Faced with such complexity, mathematicians often resort to Monte Carlo simulation to approximate the distribution. A Monte Carlo simulation is one which uses a large number of random numbers to solve a problem and does not involve the passage of time [44: 113-14]. More specifically, a Monte Carlo simulation generates a random sample from a particular distribution and uses the sample to evaluate some measure of interest, as if the sample were experimental data from an actual problem. It repeats this process for N total samples or trials and combines the measures calculated from them to draw a conclusion or approximate a quantity. For more detail, see Law and Kelton [44: 113-114], Shooman [63: 256], Noreen [51: 43-44], or Hammersley and Handscomb [33].

Because Monte Carlo results are drawn from raw observations of random data, they have varying degrees of uncertainty. This uncertainty, however, can be made fairly negligible by collecting a large number of observations [33: 4-5]. Shooman notes that in a perfect model with perfect random numbers, the error with Monte Carlo simulation will decrease proportionally to $\frac{1}{\sqrt{N}}$, where N is the number of trials [63: 259].

Monte Carlo methods are frequently employed to estimate the significance levels of test statistics. To accomplish this, the sampling distribution of a test statistic under a given null hypothesis is approximated by Monte Carlo simulation of random samples from the distribution. With this estimate, the significance levels or critical values may be derived [51: 50,63-65]. The use of modern computational power permits the generation of tens or hundreds of thousands of samples of a given sample size from a distribution. (In this paper N will denote the number of samples drawn, while n specifies the size of each of the N samples) For these samples, one can calculate an equivalent number of test statistics so that by the sheer number, their sampling distribution closely resembles the population distribution. It is in essence a "brute-force" approach to determining the sampling

distribution. And, as Aho, Bain and Engelhardt note, the simulated critical values tabled by this approach must be utilized with judgement, particularly if high accuracy is needed [2: 224].

Although some asymptotic distributions for EDF statistics have been derived, most modern goodness-of-fit tests have utilized these Monte Carlo techniques. Noreen has affirmed the utility of this approach:

> In general, a valid Monte Carlo significance level can be computed for any test statistic that is a function of data drawn from any specified population. The population does not have to have a familiar, well-behaved distribution studied by statisticians; the population can be entirely arbitrary [51: 49].

The number of samples generated in Monte Carlo studies has largely been a function of available computing power. As computers have become more capable, the scope of Monte Carlo simulations has been expanded. The benefit of larger simulation runs is better approximations of critical values for the test statistic being examined. Some typical simulation sizes for some recent Weibull goodness-of-fit studies are given below:

Author	Date of Study	Number of Trials	Reference
Crown	1991	5000	[20]
Yucel	1993	10000	[79]
Wozniak	1990	10000	[78]
Aho	1983	20000	[1]
Onen	1994	50000	[53]

For the purpose of this study, 100,000 trials will be generated for each sample size of interest to generate the critical values for each of the test statistics, $\sqrt{b_1}$ and b_2, for a Weibull distribution with known shape.

3.2.2 Plotting Positions. The mechanics of the procedure to determine critical values involves plotting a piecewise linear approximation to the CDF of the test statistic and employing linear interpolation to extract the desired critical value at a given significance level. To construct this representation, one plots the order statistics of the sample of test statistic values generated by Monte Carlo simulation on the abscissa versus a particular plotting position on the ordinate that represents its cumulative probability. These plotting position values are similar to those of

the EDF at a given order statistic and are bounded on $[0, 1]$. Hence, one can extract the critical value for, say the 95% level, by linearly interpolating between the order statistics whose plotting positions bound the value 0.95.

A more seemingly obvious means to find the critical value would be to simply extract the desired percentiles from the set of ranked sample values. In other words, in a sample of 10,000, if one wanted the 95th percentile, he would chose the 9,500th order statistic. The problem with this methodology is that it assumes the range of the test statistic is bound by the first and last order statistic, which is probably not the case. It is for this reason that the plotting position technique is utilized.

Numerous plotting position approaches have been developed in the last several decades, each of which have utility for particular applications. The major differences arise in cases with small sample sizes. Harter has conducted an extensive review of these plotting positions and concludes that the optimal choice depends on how the results of the plot will be used and on the underlying distribution of the sample [36: 1618].

Since the EDF of a sample of size N can be depicted as a step function which jumps from $\frac{(i-1)}{N}$ to $\frac{i}{N}$ at the ith order statistic of the sample (as seen if Figure 2.6), a plotting position in this range is typically used. The mean plotting position represents the expected value of the CDF at the ith order statistic of a sample size N, and is given by

$$\frac{i}{N+1}$$

The modal plotting position is computed by

$$\frac{(i-1)}{(N-1)}$$

Another popular plotting position is the median ranks, or median plotting position, which may be approximated by with the expression below: [36: 1616-17]

$$\frac{(i-0.3)}{(N+0.4)} \tag{3.1}$$

Figure 3.1 illustrates the differences. It depicts the true EDF of a sample and the step functions formed using the various plotting positions described above. Harter notes that as sample size increases beyond $N > 20$, the differences in plotting positions become negligible, but recommends the median plotting position [36: 1624]. Table 3.1 illustrates the differences between the plotting positions as sample size increases. The median ranks plotting position will be utilized here for determining the critical values for the sample skewness and kurtosis.

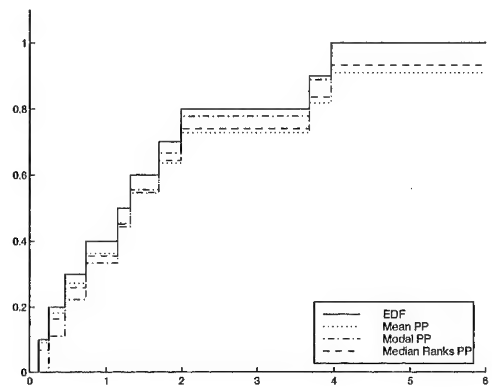


Figure 3.1 Various Plotting Positions and the EDF

Table 3.1 Comparision of Plotting Positions for $i = 2$

Plotting Position	$N = 5$	$N = 25$	$N = 50$
EDF $F_n(X_{(2)})$	0.400	0.080	0.040
Mean	0.333	0.077	0.039
Modal	0.250	0.042	0.020
Median Ranks	0.315	0.067	0.034
EDF $F_n(X_{(1)})$	0.200	0.040	0.020

Thus, to find the 90th percentile, for example, one would identify the median rank value that falls just below 0.90 and its associated ith order statistic. By necessity, the median plotting position corresponding to the $(i+1)$st order statistic will be greater than or equal to 0.90. Hence,

the critical value at the $\alpha = 0.10$ level is found by linear interpolation between the ith and (i+1)st ordered sample values. So, if y_i is the median rank position for X_i in the sample and y_{i+1} is that for X_{i+1}, then the slope of the line joining the two points is given by

$$m = \frac{y_{i+1} - y_i}{X_{i+1} - X_i}$$

and the intercept is expressed by

$$b = y_i - mX_i$$

Then, finding the critical value for the 100(1-α)% level is simply a matter of solving for x in

$$y = mx + b$$

when $y = (1 - \alpha)$. It follows that

$$\text{critical value} = \frac{(1 - \alpha) - b}{m}$$

In cases where X_i and X_{i+1} are identical, interpolation is not necessary, and the critical value is simply X_i.

In several earlier studies (e.g. Crown [20] and Bush [12]), the plotting positions 0 and 1 were assigned to x-values extrapolated from the sample, consistent with the concept that the order statistics do not define the bounds of the range of the test statistic. Given the large sample sizes generated for this project, however, and the fact that the desired critical values will not exceed the 99.5% level or fall below the 0.5% level, this extrapolation is unnecessary. To illustrate, the 100,000th order statistic from a sample will correspond to the median rank position 0.999993, a level this study will never be concerned with.

3.2.3 The Monte Carlo Procedure. The following steps outline the Monte Carlo approach used here to determine critical values for the two test statistics utilized in this sequential test, $\sqrt{b_1}$ and b_2. The procedure is illustrated by the flowchart in Figure 3.2.

1. For each value of the Weibull shape parameter $\beta = 0.5(0.5)4$, and for sample sizes $n = 10(5)50$, generate a sample of size n from the Weibull(β,1,0) distribution. The values for the location and scale parameters are chosen for convenience since the test statistics $\sqrt{b_1}$ and b_2 are location and scale invariant. Hence, the critical values generated from this particular distribution will apply for all values of location and scale.
2. Calculate the sample skewness ($\sqrt{b_1}$) and sample kurtosis (b_2) for the given sample.
3. Repeat steps (1) and (2) 100,000 times to generate a sample of $N = 100,000$ values for each of the test statistics $\sqrt{b_1}$ and b_2.
4. Order each of the statistics in an array.
5. Calculate the median rank plotting position given in Equation (3.1) for each ordered value.
6. For each significance level $\alpha = 0.005(.005)0.10$ and $0.10(0.01)0.20$ (lower tail) and $\alpha = 0.80(0.01)0.90$ and $0.90(.005)0.995$ (upper tail), use linear interpolation to find the corresponding critical values for both $\sqrt{b_1}$ and b_2. Critical values for both tails of sampling distributions will be needed since each test will be a two-sided test. The rationale for the fine granularity in the significance levels will become apparent shortly. The result at this point will be a table of critical values at the levels noted for a specified Weibull shape parameter and sample size.
7. Increment the sample size and repeat the process for $n = 10(5)50$.
8. Increment the Weibull shape parameter and repeat the process for $\beta = 0.5(0.5)4$.

3.2.4 Implementation in MATLAB. The Monte Carlo procedure described above was implemented in the MATLAB programming language, providing concise, powerful coding tools

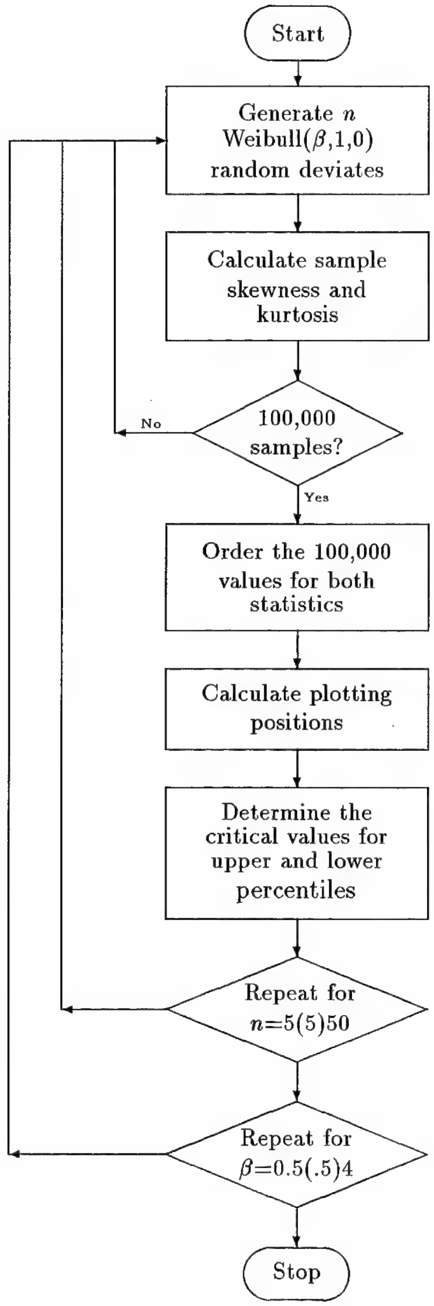


Figure 3.2 Flowchart for Critical Values Procedure

for computationally intensive routines such as this. The program was executed under MATLAB 5 on a UNIX-based Sun SPARC-20 workstation and required roughly 12 hours to run for each value of shape. Each run generated a total of 27,500,000 Weibull deviates of each given shape ($N = 100,000$ samples of sizes $n = 5(5)50$). The Weibull random deviates were generated using the **weibrnd** function provided in the MATLAB Statistics Toolbox, which employs the inverse transformation method to produce its pseudo-random variates. The powerful matrix-handling capabilities of MATLAB were employed by representing each sample of size n as an 1 x n array and manipulating the samples and their test statistics in that form. The code for this procedure can be found in Appendix H.1, and the critical values found with this procedure are provided in Appendix A

3.3 Formal Statement of the Tests

Now that the critical values for the two test statistics have been generated with Monte Carlo simulation, the individual skewness and kurtosis tests can be formally presented.

3.3.1 Skewness Test. Given a random sample $X_1, X_2, ..., X_n$, and hypothesized Weibull shape β, the sample skewness test can be summarized as follows:

$\mathrm{H}_0 : X \sim$Weibull (β)

$\mathrm{H}_a : X \not\sim$Weibull ($\beta$)

Test Statistic:

$$\sqrt{b_1} = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^3}{\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2\right]^{\frac{3}{2}}} \tag{3.2}$$

For a significance level α, the rejection region is given by

$$\sqrt{b_1} > \sqrt{b_1}_{(1-\frac{\alpha}{2})} \quad \text{or} \quad \sqrt{b_1} < \sqrt{b_1}_{\frac{\alpha}{2}}$$

where the values of $\sqrt{b_1}_{\frac{\alpha}{2}}$ (lower tail) and $\sqrt{b_1}_{(1-\frac{\alpha}{2})}$ (upper tail) can be found in the tables provided in Appendix A for sample size n and shape β.

Note that the two-sided nature of the test requires critical values in increments of $\frac{\alpha}{2}$; hence, the need for increments of 0.005 in the α-levels of the previous simulation.

3.3.2 Kurtosis Test. Given a random sample $X_1, X_2, ...X_n$, and hypothesized Weibull shape β, the sample kurtosis test can be summarized as follows:

$\mathrm{H}_0 : X \sim$Weibull (β)

$\mathrm{H}_a : X \nsim$Weibull (β)

Test Statistic:

$$b_2 = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^4}{\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2\right]^2} \tag{3.3}$$

For a significance level α, the rejection region is given by

$$b_2 > b_{2(1-\frac{\alpha}{2})} \quad \text{or} \quad b_2 < b_{\frac{\alpha}{2}}$$

where the values of $b_{2\frac{\alpha}{2}}$ (lower tail) and $b_{2(1-\frac{\alpha}{2})}$ (upper tail) can be found in the tables provided in Appendix A for sample size n and shape β.

3.3.3 The Sequential Test. The sequential test proposed in this paper consists simply of combining the two tests above and applying one after another, in any order, to the same sample. If a given sample fails either test, the null hypothesis is rejected. A sample passes the test (fails to reject H_0) only if the sample passes both tests at their selected significance levels. The test may be formally stated as follows:

Given a random sample $X_1, X_2, ...X_n$, and hypothesized shape β

$\mathrm{H}_0 : X \sim$Weibull (β)

$H_a : X \not\sim$ Weibull (β)

Test Statistics: $\sqrt{b_1}$ and b_2 as given in (3.2) and (3.3)

For an overall significance level α, the rejection region is given by

$$\sqrt{b_1} > \sqrt{b_1}_{(1-\frac{\alpha_1}{2})} \quad \text{or} \quad \sqrt{b_1} < \sqrt{b_1}_{\frac{\alpha_1}{2}}$$

OR

$$b_2 > b_{2(1-\frac{\alpha_2}{2})} \quad \text{or} \quad b_2 < b_{2\frac{\alpha_2}{2}}$$

where α_1 and α_2 are the selected significance levels of the individual tests that yield the overall attained significance level of α for the sequential test. A host of such combinations exists, and it is up to the user to select them based on the overall level he desires to achieve. It is the determination of the relationship between α and the individual levels, α_1 and α_2 that we focus on next.

3.4 Attained Significance Levels

3.4.1 Background. If the skewness test is conducted at some significance level α_1, and the kurtosis test at a level α_2, then Bonferonni's Inequality informs us only that the overall significance level of the sequential test, α, is bounded above by $\alpha_1 + \alpha_2$. To make the sequential test useful, one needs to be able to identify a value for α, given α_1 and α_2, or to chose α_1 and α_2 appropriately to attain a desired overall level α. As noted, various combinations of α_1 and α_2 may yield very similar values for α, permitting the user to opt for particular levels of the skewness and kurtosis test based on the alternatives to be considered. This facilitates some degree of directionality in the tests. For example, if one is primarily concerned with greater power in distinguishing among skewed alternatives, he could elect to use a higher significance level (larger α_1 value) in the skewness test and find the appropriate value for α_2 to achieve an overall desired level α. Since increasing Type I error increases power, this technique would improve the power of the skewness test and still

maintain the same overall significance level. Likewise, if accuracy in the tails was a key concern, using a higher significance level in the kurtosis test might be appropriate.

Monte Carlo simulation is again the tool of choice for empirical determination of these attained significance levels for the sequential test. Both Onen and Gunes have successfully used this approach, and a variation on their procedure will be employed here [31] [53]. Initially, the approach is similar to that used in the determination of critical values; however, this procedure will actually conduct the two tests using the critical values found in the previous simulations. As before, a large number of Weibull samples of a given size are generated, and the test statistics for each calculated. Each sample then, will be subjected to the sequential skewness and kurtosis tests at all possible combinations of significance levels, α_1 and α_2, to determine if it passes or fails. Recall if it fails just one of the tests, it fails the overall test. By counting the number of samples that fail at each combination of levels, one can evaluate the attained significance levels for the sequential test. Since α is simply the probability of rejecting a true null hypothesis, and the samples generated are indeed Weibull samples, α can be found by calculating the percentage of all samples that are rejected at a given combination of levels α_1 and α_2. Since we are concerned with cases of known shape and various sample sizes, the attained significance levels will have to be established for each sample size and shape combination.

The algorithm developed by Onen to implement this procedure on a computer involved counting the number of samples that passed both tests and storing the results in an array structure indexed by the significance levels. For example, in an array A, the element A_{ij} held the count of samples passing both tests at level $\alpha_1 = \frac{i}{100}$ for the first test and $\alpha_2 = \frac{j}{100}$ for the second test. Instead of generating new samples for each increment in the α-values, he simply subjected each of the 50,000 samples to all levels of each test from α =0.01(.01)0.20. So, each sample was tested some 40 times (20 levels for each of the two tests). The flowchart in his text seems to indicate he actually tested each sample at every possible combination of significance levels for the two tests,

meaning 400 tests on each sample. A closer look at his actual code, however, reveals that he actually only conducted 40 tests on each sample, as noted above. After all 50,000 samples were tested, the array structure contained the number of samples that passed the sequential test at each of the 400 combinations of significance levels for the two tests. Onen found the proportion of samples that incorrectly failed the test by dividing each element in the array by 50,000 and then taking the complement of these values [53].

3.4.2 A Revised Approach. A slight variation of Onen's approach was implemented for this effort. In this implementation, the counter array A tracks the number of samples *failing* the sequential test (failing one or both of the individual tests). This simplified the coding logic and was more direct. Additionally, by applying some simple results from hypothesis testing the number of tests each sample is subjected to was reduced moderately.

If a given sample fails any hypothesis test at a level α, say $\alpha = 0.01$, then it will obviously fail the test at any higher significance level, such as $\alpha = 0.05$, because increasing α essentially enlarges the rejection region for the test. It is this truth that aides our effort. If a test presented here is conducted starting with the lowest significance level and working toward higher levels (smaller to larger α levels), then as soon as a sample fails at a particular level, it follows that it also fails at every subsequent level. So, the test does not actually have to be executed any further. Also simplifying matters here is the fact that in a sequential test the sample fails once it fails just one of the member tests. Hence, once a sample fails at a given level for the first test, it will fail for every other larger α-value for that test, and the results from the second test are irrelevant – the sample has still failed the overall test. So, once a failure level is determined for one test, one may conclude that the sample will fail for all remaining (larger) α-levels for the given test and all levels of the other test.

Consequently, there are four possible outcomes to the application of the sequential test at all combinations of significance levels for the two member tests. Tables 3.2 to 3.4, depicting possible

results of two tests over a range of α-levels between 0.01 and 0.10, help illustrate the point. The Pass-Fail tables below actually reflect the results for a single sample in the counter array A mentioned above. A "-" indicates passing the sequential test at the given combination of levels, and an "X" indicates failure. The four outcomes are as follows:

1. The given sample could pass both tests at all levels; hence, no elements in the counter array A would be incremented.

2. The sample could fail Test #1 at some level, say $\alpha_1 = 0.05$ and pass Test #2 at all levels. Here, the sample would fail the sequential test at levels for $\alpha_1 \geq 0.05$ across all levels of α_2. This case is illustrated in Table 3.2.

Table 3.2 Pass-Fail Table: Fail Test #1 Only at α=0.05

Test #1	Test #2 α-level									
α-level	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
0.01	-	-	-	-	-	-	-	-	-	-
0.02	-	-	-	-	-	-	-	-	-	-
0.03	-	-	-	-	-	-	-	-	-	-
0.04	-	-	-	-	-	-	-	-	-	-
0.05	X	X	X	X	X	X	X	X	X	X
0.06	X	X	X	X	X	X	X	X	X	X
0.07	X	X	X	X	X	X	X	X	X	X
0.08	X	X	X	X	X	X	X	X	X	X
0.09	X	X	X	X	X	X	X	X	X	X
0.10	X	X	X	X	X	X	X	X	X	X

3. The sample could pass Test #1 at all levels and fail Test #2 at some level, for instance $\alpha_2 = 0.05$. The result is parallel to that in (2), and is shown in Table 3.3. The sample fails the sequential test for all combinations in which $\alpha_2 \geq 0.05$ and all values of α_1.

4. The sample could fail both tests at some level. For example, consider a sample that fails Test #1 at $\alpha_1 = 0.07$ and fails Test #2 at $\alpha_2 = 0.05$. The status of passes and fails in the sequential test corresponds to Table 3.4. The sample fails the sequential test at all combinations where $\alpha_1 \geq 0.07$ or $\alpha_2 \geq 0.05$.

Table 3.3 Pass-Fail Table: Fail Test #2 Only at α=0.05

Test #1	Test #2 α-level									
α-level	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
0.01	-	-	-	-	X	X	X	X	X	X
0.02	-	-	-	-	X	X	X	X	X	X
0.03	-	-	-	-	X	X	X	X	X	X
0.04	-	-	-	-	X	X	X	X	X	X
0.05	-	-	-	-	X	X	X	X	X	X
0.06	-	-	-	-	X	X	X	X	X	X
0.07	-	-	-	-	X	X	X	X	X	X
0.08	-	-	-	-	X	X	X	X	X	X
0.09	-	-	-	-	X	X	X	X	X	X
0.10	-	-	-	-	X	X	X	X	X	X

Table 3.4 Pass-Fail Table: Fail Both: Test #1 at α=0.07 and Test #2 at α=0.05

Test #1	Test #2 α-level									
α-level	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
0.01	-	-	-	-	X	X	X	X	X	X
0.02	-	-	-	-	X	X	X	X	X	X
0.03	-	-	-	-	X	X	X	X	X	X
0.04	-	-	-	-	X	X	X	X	X	X
0.05	-	-	-	-	X	X	X	X	X	X
0.06	-	-	-	-	X	X	X	X	X	X
0.07	X	X	X	X	X	X	X	X	X	X
0.08	X	X	X	X	X	X	X	X	X	X
0.09	X	X	X	X	X	X	X	X	X	X
0.10	X	X	X	X	X	X	X	X	X	X

3.4.3 Algorithmic Implementation. Given this reality, the pass/fail results for each combination of the two tests can be determined without actually testing them all. This is done by simply finding the first failure level of each test individually and then using the facts above to determine which combinations result in failure for the sequential test. This procedure can be visualized as testing "down" column 1 with Test #1 and "across" row 1 with Test #2 without considering the other test's level, in any of the figures above. Once the failure point is found for each test, one may "fill the squares" for failures for the other combinations as noted above without having to conduct each sequential test. Thus, only 20 tests would be needed, vice 100, for α-levels examined between 0.01 and 0.10 in the examples above.

The above procedure, as implemented in this research, can be summarized as follows. The flowchart in Figure 3.3 illustrates it. Since the skewness or kurtosis tests can be applied in any order, they are generically referred to as Test #1 and Test #2.

1. Begin as before by selecting a shape parameter β and sample size n for the Weibull samples according to the specific null hypothesis being investigated. Generate n Weibull$(\beta, 1, 0)$ random deviates.

2. Calculate the test statistics $\sqrt{b_1}$ and b_2 for the sample.

3. Initialize counters to track the current levels of the two tests; i_{curr} for Test #1 and j_{curr} for Test #2. The significance levels are $\alpha_1 = i_{curr}/100$ and $\alpha_2 = j_{curr}/100$. Start with both i_{curr} and $j_{curr} = 1$. Another pair of indices will indicate the level of the first failure for each test; i_{stop} for Test #1 and j_{stop} for Test #2.

4. Conduct the first test (Test #1) on the sample for $\alpha_1 = i_{curr}/100$. If the sample fails the test at this level, record the current level in i_{stop}. Then proceed to Step (6).

5. If the sample passes at the current level, increment i_{curr} by 1. If the range of desired levels has been tested ($i_{curr} > 0.20$) then leave $i_{stop} = 0.21$, indicating no failures, and proceed to Step (6). Otherwise, return to Step (4) with the new value for i_{curr}.

6. Conduct the second test (Test #2) on the sample for $\alpha_2 = j_{curr}/100$. If the sample fails the test at this level, record the current level in j_{stop}. Then proceed to Step (8).

7. If the sample passes at the current level, increment j_{curr} by 1. If the range of desired levels has been tested ($j_{curr} > 0.20$) then leave $j_{stop} = 0.21$, indicating no failures, and proceed to Step (8). Otherwise, return to Step (6) with the new value for j_{curr}.

8. Now that the failure points have been determined, increment the appropriate counters in the array A. Specifically, increment A_{ij} for all (i,j) such that $i \geq i_{stop}$ or $j \geq j_{stop}$, avoiding duplication in the intersection of the two sets.

9. Repeat Steps (1) through (8) for 100,000 samples.

10. When finished, the array element A_{ij} will hold the counts for number of failures (rejection of the true null hypothesis) for the corresponding combinations of significance levels $\alpha_1 = i/100$ and $\alpha_2 = j/100$. To find the attained significance level for a given combination, α_{ij}, simply divide A_{ij} by the total number of samples, 100,000.

The technique employed by Onen was, as noted, similar to that described here; however, his procedure counted passes and performed each test at all 20 α-levels regardless of whether a failure point was reached. A potential flaw in his code emerged at this point. The flag variable he utilized to identify the α-level where the test initially failed was not reset between samples, creating a problem if a given sample fails at the lowest significance level (0.01). In this case, the failure flag is not assigned, thus retaining its value from the previous sample tested. Consequently, the new sample is counted as passing at the same levels as the prior one, resulting in the counter array being incremented for numerous combinations of levels which the sample actually failed to pass. The outcome is an underestimation of the attained significance levels and of the power for the test when he used similar code for his power study. This problem only arises, however, if a sample fails at the 0.01 level, not if it passes. The revised algorithm employed in the current effort rectifies this shortfall. Another key difference in this new sequential procedure is the fact that both tests are two-sided, thereby mandating the aforementioned fine granularity in the levels of the critical values. The results of these attained significance level calculations are presented in Appendix B and discussed in the following chapter.

3.5 Power Study

The key measure of effectiveness for a goodness-of-fit test is its power — the probability it will properly reject the null hypothesis when it is not true. Once the attained significance levels have been determined for the new test, it is possible to evaluate its power against particular alternate

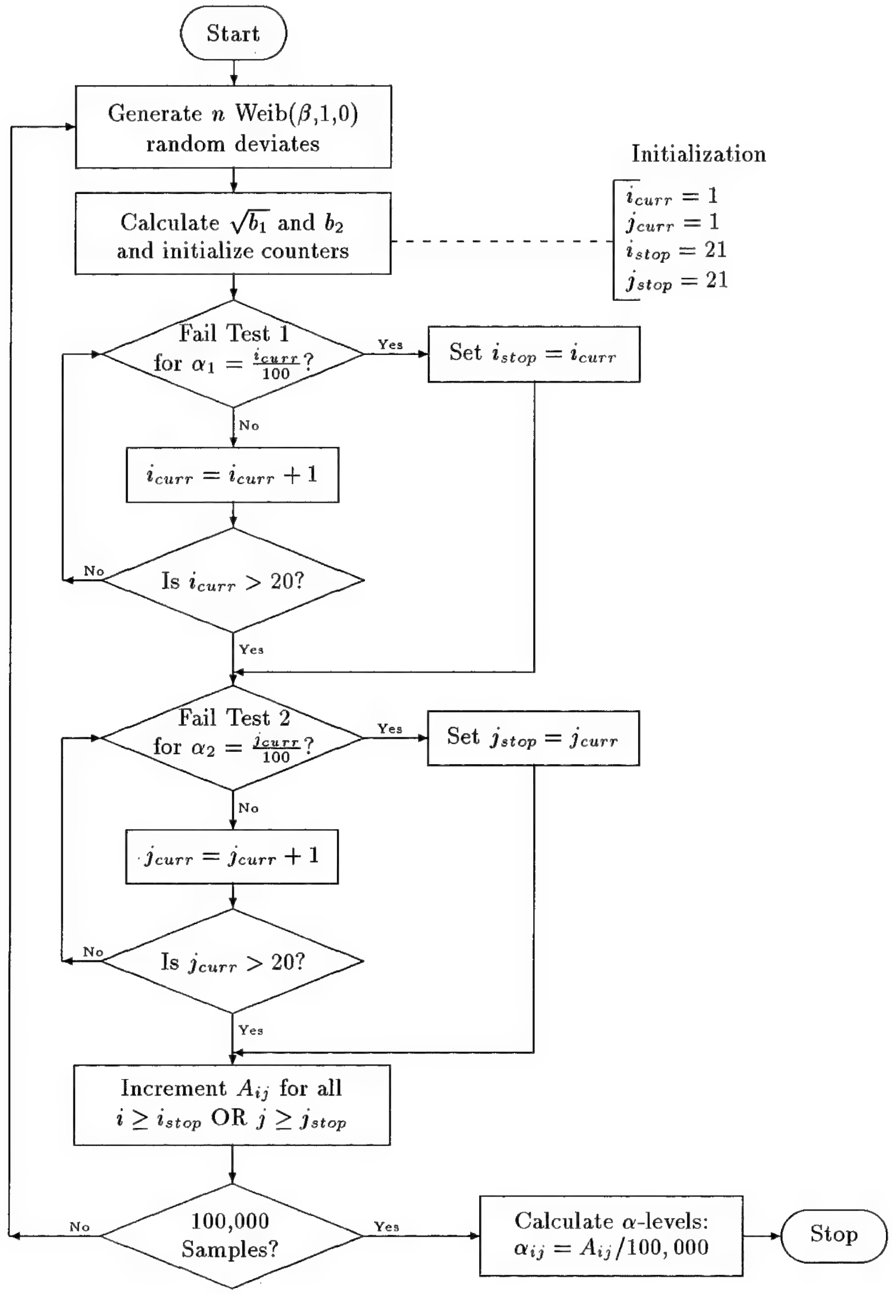


Figure 3.3 Flowchart for Attained Significance Levels Procedure

hypotheses to determine if it indeed demonstrates any improved capability compared to existing procedures. As discussed in Chapter 2, this study uses the popular and powerful Cramér-von Mises and Anderson-Darling EDF tests for this comparative analysis. The even more widespread but less powerful Kolmogorov-Smirnov EDF test is also utilized in this power study because it is heavily documented as well. If this new sequential test can demonstrate roughly equivalent or better power than these existing tests across a broad range of alternate distributions, it would represent a significant contribution to the goodness-of-fit field. Because this test requires no complex parameter estimation routines, it could serve as a much more computationally efficient means of conducting adequacy of fit tests for the Weibull distribution.

To facilitate a comparative power analysis, the structure of this power study was based on two existing Weibull power studies using EDF statistics — one by Bush [13] and another by Wozniak [77]. The alternate distributions and sample sizes considered in this research were selected to correspond with those from these earlier studies, enabling a comparison of the results. Consequently, this power study is subdivided into a portion addressing the Bush alternatives and another investigating those Wozniak examined. Since Wozniak tested for the two-parameter Weibull using an extreme-value test, her results are not directly comparable to those found in this sequential test. Nonetheless, the alternates she examined appear frequently in the literature, and her results provide a general context for evaluating the results found in this study. As part of a systematic approach, this study evaluated the sequential test as well as the two component tests individually. Looking at the results in concert yields clues indicating the better of the two tests against particular alternatives and insights into the complementary role they play as partners in the omnibus sequential test. The investigation also considered directional versions of the separate skewness and kurtosis tests by employing them in a one-sided instead of two-sided manner. Since a one-sided test enlarges the rejection region in the tail where a potential discrepancy is expected, it should improve the power of the test, presuming enough information exists to justify a one-sided test. If so, using one or both directional tests in a sequential fashion could be a means of improving its

power. Provided one believes, for instance, that if a given sample is not Weibull with shape $\beta = 1$, that it will be more highly skewed ($\sqrt{\beta_1} > 2$), he could use the directional upper tail skewness test in the sequential procedure and expect to find higher discriminatory ability. Consequently, several directional power studies were accomplished to quantify what degree of improvement would result from using one-sided tests for skewness and kurtosis separately and in the sequential test. D'Agostino and Stephens discuss the improved power of such directional variants of the $\sqrt{b_1}$ and b_2 tests relative to the very similar R-test for normality and as stand-alone tests [69: 403].

3.5.1 Monte Carlo Procedure. The implementation of the power study is nearly identical to the Monte Carlo algorithm used in determining the attained significance levels, with only slight changes. That effort estimated the proportion of times the test incorrectly rejected a true null hypothesis by generating Weibull samples and comparing the test statistics to the appropriate critical values for that particular Weibull distribution. For the power study, in contrast, the goal is to measure the proportion of times the test correctly rejects samples from distributions other than that in the null hypothesis. Hence, instead of generating Weibull samples with a shape specified in that null hypothesis, the data generated in the power study simulation will come from specific alternate distributions. The sample skewness and kurtosis values, though, are compared as before to the critical values for the null Weibull hypothesis. Since each of the two tests can be evaluated at various combinations of significance levels for a given level of the overall test, one must once again determine the rejection rate (power) for each of these combinations. It is clear, then, that the basic algorithm used in the attained significance level study will serve our purposes here by simply changing the generator for the samples being tested.

Thus, to determine the power of the sequential test in distinguishing between a sample of size n from a Weibull distribution with shape β (H_0) and that of some specified alternate distribution (H_a), repeat the steps for the attained significance level algorithm with the exception of step (1), which is replaced by:

1. Generate n random deviates from the alternate distribution.

The end product of the procedure will be an array of powers for each of the possible combinations of skewness and kurtosis tests at levels $\alpha = 0.01(.01)0.20$.

Two other slight modifications to the algorithm are appropriate and will reduce the run times of the simulation. First, instead of calculating powers for all sample sizes $n = 5(5)50$ as before, the only sample sizes studied will be those that were examined by Bush or Wozniak for the particular alternatives they considered. Second, since there is not as pressing a need for accuracy in power estimates as there was for critical values and attained significance levels, the power study does not require 100,000 samples to be generated. Instead, only 40,000 samples were used. These two changes reduced run times from in excess of 12 hours to 1-5 hours.

The justification for reducing the size of the Monte Carlo runs is straight-forward. Since the objective of the power study is to generate points of comparison with other published tests, accuracy in the estimates is not as critical. Correct estimates in just the first two decimal places are deemed sufficient; thus, as long as they fall within ± 0.005 of the true values, they are considered valid. Power is fundamentally a proportion, and estimation theory informs us that the standard error of an estimate of a proportion, $\widehat{p}$ is: [71: 326]

$$\sigma_{\widehat{p}} = \sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}. \tag{3.4}$$

A rough 95% confidence interval about p can be formed by $\widehat{p} \pm 2\sigma_{\widehat{p}}$, according to the empirical rule [71: 9]. The objective then is to ensure n is large enough so that the half-width $2\sigma_{\widehat{p}} \leq 0.005$. Also note that $\sigma_{\widehat{p}}$ is maximized when $\widehat{p} = \frac{1}{2}$; thus, by substitution in the expression for $\sigma_{\widehat{p}}$ and solving for n, one finds that $n \geq 40,000$ will keep the maximum half-width of the confidence interval within the criteria specified. This rationale led to the use of 40,000 samples vice 100,000 in the power study.

3.5.2 Alternate Distributions. Numerous alternate distributions were evaluated in this power study, and for comparative purposes they are nearly identical to those in the EDF studies of Bush and Wozniak.

3.5.2.1 The Bush Alternatives. In his study, Bush compared the Weibull distributions with shape $\beta = 1$ and 3.5 with sample sizes $n = 5, 15,$ and 25 against the following alternate distributions:

- Weibull(β=1)
- Weibull(β=2)
- Weibull(β=3.5)
- Beta(2,2)
- Beta(2,3)
- Gamma(2,1)
- Normal(10,1)
- Uniform(1,2)

This investigation uses the same null hypotheses and alternates with a few minor variations. Since the uniform and normal are location and scale-parameter distributions, and this sequential test is location and scale invariant, the parameter values are irrelevant. For convenience, the Uniform(0,2) and Normal(0,1) were used instead. Additionally, the sample size 50 was examined to quantify the power at larger sample sizes.

The use of the Weibull shape $\beta = 1$ alternative against the same null hypothesis, and the similar pairing for $\beta = 3.5$ serves as a means to verify the simulation code and validate the accuracy of the critical values. The powers in these two cases should be nearly identical (within some small random error) to the attained significance levels calculated in the previous simulation for a given shape and sample size. If so, the critical values, attained significance levels, and the simulation code for all three facets of this research can be considered verified.

The functional forms of the density functions for the non-Weibull alternate distributions are given in Figures 3.4 to 3.7 [44: 330-339]. To generate the random variates from each of the above distributions the appropriate routines in the MATLAB Statistics Toolbox were utilized. The specific MATLAB functions used and the generation method they employ are shown in Table 3.5

(For a thorough explanation of the various techniques used in random variate generation, see Law and Kelton [44: 462-493]).

Table 3.5 MATLAB Random Variate Generators

Distribution	MATLAB Function	Generation Method [70] [49]
Weibull	weibrnd	Inverse Transformation
Gamma	gamrnd	Accept-Reject or Inverse Transform (depending on shape α)
Uniform	unifrnd	Lagged Fibonacci and shift register generator
Normal	normrnd	Based on work of George Marsaglia
Beta	betarnd	Ratio of gamma deviates

3.5.2.2 The Wozniak Alternatives. The distributions studied by Wozniak were considerably different and demanded considerably more work for any type of comparison between this moment-based test and her EDF test results. Specifically, she examined the following distributions for sample sizes $n = 20$ and 30: [77: 152]

- Chi-squared with 1 degree of freedom $\left(\chi^2(1)\right)$
- Chi-squared with 4 degrees of freedom $\left(\chi^2(4)\right)$
- Transformed Double Exponential
- Lognormal(0,1)
- Transformed Logistic(0,1)
- Transformed Cauchy(0,1)

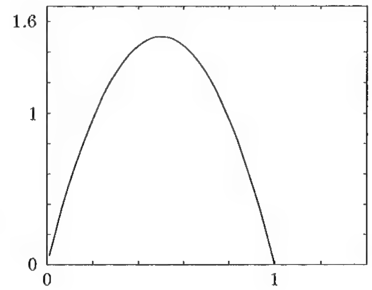
These particular alternatives are commonly used in power studies to evaluate goodness-of-fit tests for the Weibull distribution. A key factor that distinguishes her research from the moment-based approach here, is that it deals only with the two-parameter Weibull null hypothesis (no location parameter) and therefore addressed the goodness-of-fit problem by testing for the extreme-value distribution on log-transformed data from the distributions noted above. Her test, then, was fundamentally different. This fact creates two problems — identifying the Weibull shape to use as the null hypothesis and generating samples from some of the distributions she used.

To recognize the first problem, one must note that the log-transformation that converts the two-parameter Weibull data to extreme-value data converts the Weibull shape and scale parameters to scale and location parameters respectively. If β and θ are the Weibull shape and scale, then $\delta = 1/\beta$ and $\xi = \log(\theta)$ are the scale and location parameters in the resulting extreme value distribution [77: 153]. The EDF tests, of course, require estimation of the location and scale

$f(x) = \frac{x^{\alpha_1-1}(1-x)^{\alpha_2-1}}{B(\alpha_1,\alpha_2)}$ for $0 < x < 1$

where $B(\alpha_1, \alpha_2) = \int_0^1 t^{\alpha_1-1}(1-t)^{\alpha_2-1}dt$

with $\alpha_1 = 2$ and $\alpha_2 = 2$ the pdf has this form.



With $\alpha_1 = 2$ and $\alpha_2 = 3$ the pdf has this form.

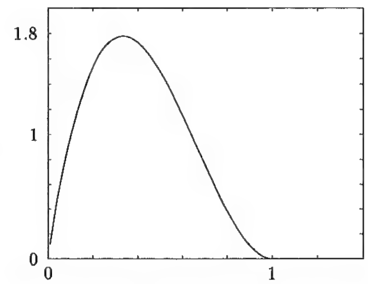


Figure 3.4 Beta(α_1, α_2), shape parameters $\alpha_1 > 0$ and $\alpha_2 > 0$

$f(x) = \frac{\beta^{-\alpha}x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)}$ for $x > 0$

With $\alpha = 2$ and $\beta = 1$ the pdf has this form.

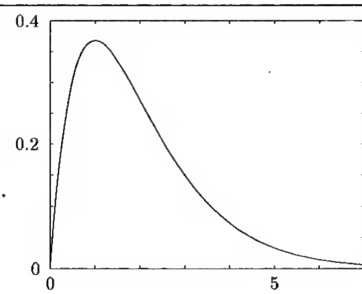


Figure 3.5 Gamma(α, β), shape parameter $\alpha > 0$ and scale $\beta > 0$

$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$ for $x \in R$

With $\mu = 0$ and $\sigma = 1$ the pdf has this form.

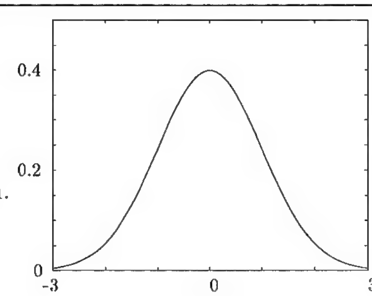


Figure 3.6 Normal(μ, σ), location $\mu \in (-\infty, \infty)$ and scale $\sigma > 0$

$f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$

With $a = 0$ and $b = 2$ the pdf has this form.

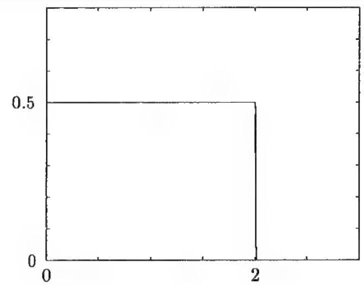


Figure 3.7 Uniform(a, b) $a, b \in R$, $a < b$ (a is location, $b - a$ is scale)

parameters, meaning that, in essence, Wozniak's procedures estimated the related Weibull shape and scale parameters. Since the sequential test developed in this study is predicated on known shape values, a direct comparison with her results is not possible. Nevertheless, some comparison is possible using the known shape values in the range studied that are close to those that her procedure would have essentially estimated. To determine these values, 100 samples of sizes $n = 20, 30$ were generated from each of her alternate distributions. For each of these samples, the MLE for the Weibull shape parameter was calculated. Examination of the means of these MLEs led to choices for the appropriate null hypotheses to use in comparing powers with her published results. The 100-sample means for each alternate and the values of shape chosen for comparison are shown in Table 3.6. MLEs for Weibull shape on $\chi^2(1)$ samples, for example, tended to fall near 0.65. Thus, the closest two Weibull shapes examined here, $\beta = 0.5$ and 1, were used as null hypotheses in power studies with the $\chi^2(1)$ alternative.

Obviously, the comparisons made in this power study will need to be interpreted in light of these findings. In many cases, the estimated shape values compare closely with the discrete values examined here; hence, the power comparisons can be taken somewhat at face value. For some, however, the estimated shapes fall between two of the discrete values, meaning the assessment of power will be somewhat more subjective. Even if both tests are identically powerful, the fact that Wozniak's results came from estimated parameters and the sequential test results from fixed discrete values would lead one to expect higher power for the sequential test where the discrepancies

Table 3.6 Shape Estimates for Wozniak's Alternates

Distribution	Sample Size	Mean Shape Estimate	Std Deviation	Shape(s) Used for H_0 Here
$\chi^2(1)$	20	0.6658	0.1263	$\beta = 0.5, 1$
	30	0.6494	0.0942	
$\chi^2(4)$	20	1.6263	0.3146	$\beta = 1.5$
	30	1.5684	0.2448	
Lognormal	20	1.1426	0.2199	$\beta = 1$
	30	1.0819	0.1872	
X Logistic	23	0.6038	0.1699	$\beta = 0.5, 1$
	30	0.5812	0.1173	
X Double Exp	20	0.8031	0.2455	$\beta = 0.5, 1$
	30	0.7616	0.2079	
X Cauchy	20	0.3467	0.5224	$\beta = 0.5, 1$
	30	0.3783	0.5473	

between the null and alternate hypotheses were exaggerated. Beyond the questionable comparative utility of the power study though, the results are nevertheless valuable as quantifiers of the test's performance against popular benchmarks for Weibull goodness-of-fit tests.

The second problem of generating samples from the distributions Wozniak used arises only for the transformed distributions. The two chi-squared distributions and the lognormal distribution are relatively easy to generate deviates from; in fact, MATLAB's Statistics Toolbox contains routines for them (**chi2rnd** and **lognrnd**). Recall that the Chi-squared distribution with k degrees of freedom is a special case of the Gamma distribution with shape $\frac{k}{2}$ and scale 2 [44: 332]. Hence, the **chi2rnd** function utilizes the **gamrnd** routine mentioned earlier. The **lognrnd** function is based on a simple exponential transformation of the **normrnd** generator output [49]. (Note that while Wozniak log-transformed the data generated by her alternate distributions, this study did not perform that transform since it tests Weibull data, not extreme-value data.) The functional forms of these distributions are shown in Figures 3.8 and 3.9 [71: 160,287].

The last three alternatives were somewhat more problematic. Instead of using the logistic, double exponential and Cauchy distributions in their standard forms defined over the entire real line, Wozniak converted the variates to the positive real line with an exponential transformation so that they would be nonnegative, like Weibull deviates are. In other words, given Y_i was a random

$f(x) = \frac{1}{2^{v/2}\Gamma(v/2)} x^{v/2-1} e^{-x/2}$, for $x > 0$

With $v = 1$ it has this form

With $v = 4$ it has this form.

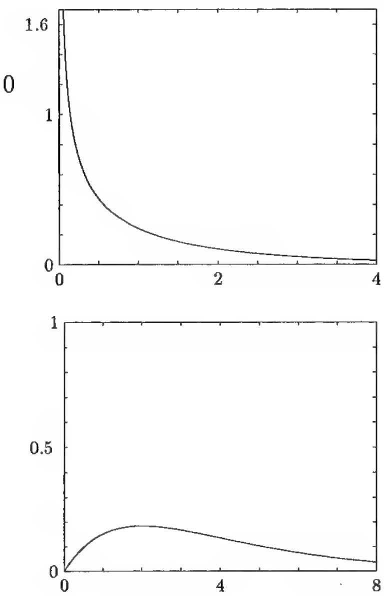


Figure 3.8 Chi-squared, $\chi^2(v)$, with degrees of freedom v

$f(x) = \left(\frac{1}{x\sigma\sqrt{2\pi}}\right) e^{-(\log x-\mu)^2/(2\sigma^2)}$, for $x > 0$

With $\mu = 0$ and $\sigma = 1$ is has this form.

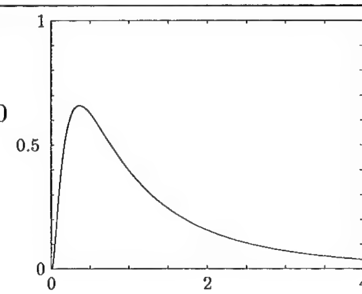


Figure 3.9 Lognormal(μ, σ), location $\mu \in (-\infty, \infty)$ and scale $\sigma > 0$

variate generated from one of these distributions, her procedure uses $X_i = \exp(Y_i)$ [77]. Hence, the X_i were tested as Weibull data, not the Y_i. Since her method actually tested the extreme-value distribution, it then log-transforms the X_is back to Y_is and tests them as extreme-value samples. The sequential test in the current study tests hypothesized Weibull data; consequently, this power study must generate X_is from the exponentially-transformed logistic, double exponential, and Cauchy samples and test them as if they originated from a Weibull population.

New random number generators for these transformed distributions had to be coded. To this end, the inverse transformation approach was used to create generators for the Logistic, Double exponential, and Cauchy distributions. By then applying the exponential function to these generators, the new generators were derived for the transformed distributions.

To get an idea of what functional forms these new distributions assumed, their PDFs were derived. An examination of their shape complements the previous determination of the appropriate Weibull shapes used in the power study. The density functions of the original distributions, the transformed version, and the generator derived for this study are shown in Figures 3.10 through 3.12. One can readily note that all three PDFs of the transformed distributions most closely resemble Weibull shapes $\beta = 0.5$ and 1, agreeing with the earlier conclusion.

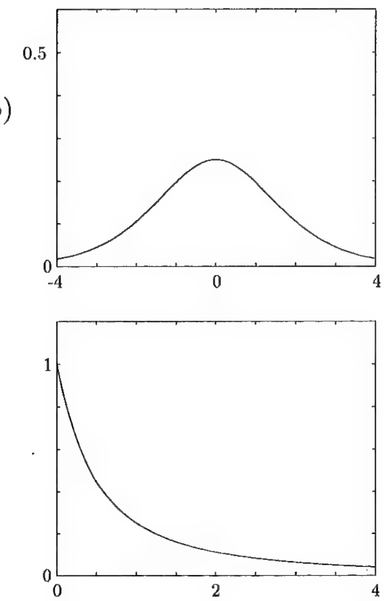
3.5.2.3 Moments of Alternate Distributions. Since the test proposed by this research is based upon sample moments, it is important to make some observations regarding the moments of the alternate distributions utilized in the power study. The theoretical skewness and kurtosis values for these alternates were derived and tabled in Table 3.7. Note that the three transformed alternatives Wozniak used do not have theoretical moments — they fail to exist. This does not, however, mean that the skewness and kurtosis tests cannot be applied because although $\sqrt{\beta_1}$ and β_2 do not exist, the sample moments used as the test statistics, $\sqrt{b_1}$ and b_2, usually do exist in these cases. In fact, for the transformed Logistic (X Logistic) and Double Exponential (X Double Exp.) alternatives, no cases were observed in which the sample moments failed to exist, although they do

$f_y(y) = \frac{\frac{1}{\beta}\exp[-(y-\alpha)/\beta]}{(1+\exp[-(y-\alpha)/\beta])^2}$ for $y \in (-\infty, \infty)$

With $\alpha = 0$ and $\beta = 1$ it has this form.

Transformed Logistic(0,1)

$f_x(x) = \frac{1}{(1+x)^2}$ for $x > 0$



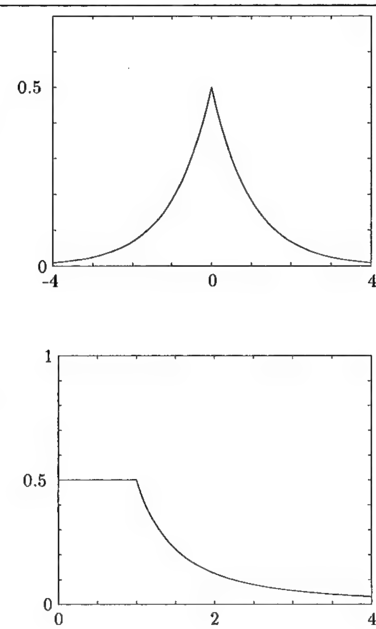
Generator: $X = \exp\left[-\log\left(\frac{1}{U} - 1\right)\right] = \frac{U}{1-U}$, where U~Uniform(0,1)

Figure 3.10 Logistic(α, β), location $\alpha \in (-\infty, \infty)$, scale $\beta > 0$

$f_y(y) = \frac{1}{2}e^{-|y|}$ for $y \in (-\infty, \infty)$

Transformed Double Exponential

$$f_x(x) = \begin{cases} 0 & if \quad x < 0 \\ \frac{1}{2} & if \quad 0 \leq x \leq 1 \\ \frac{1}{2x^2} & if \quad 1 < x \end{cases}$$



Generator:

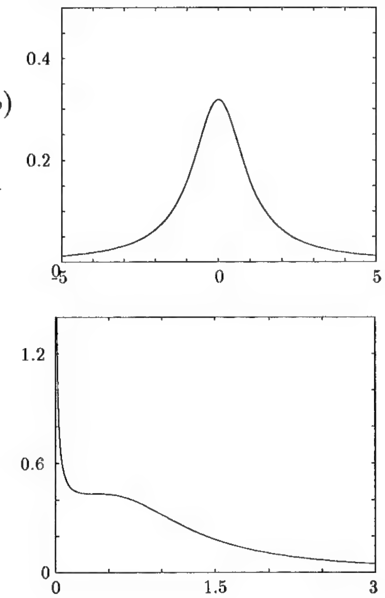
$$X = \begin{cases} 2U & if \quad U \leq 0.5 \\ \frac{1}{2-2U} & if \quad U > 0.5 \end{cases}$$

where $U \sim$ Uniform(0,1)

Figure 3.11 Double Exponential

$f_y(y) = \frac{\beta}{\pi}\left(\frac{1}{\beta^2+(y-\alpha)^2}\right)$ for $y \in (-\infty, \infty)$

When $\alpha = 0$ and $\beta = 1$ it has this form



Transformed Cauchy(0,1)

$f_x(x) = \frac{1}{\pi y(1+\ln^2 x)}$ for $x > 0$

Generator:

$X = \exp\left(\tan\left[\pi\left(U - \frac{1}{2}\right)\right]\right)$

where U~Uniform(0,1)

Figure 3.12 Cauchy(α, β), location $\alpha \in (-\infty, \infty)$, scale $\beta > 0$

exhibit a high degree of variability. For the transformed Cauchy (X Cauchy), though, the sample moments failed to exist somewhat frequently, and that frequency increased with sample size. The cause for this stems from the nature of the random variate generator. It is capable of producing values exceeding machine precision, which MATLAB reports as ∞ (Inf). Consequently, sample skewness and kurtosis for such samples cannot be calculated. Table 3.8 summarizes by sample size the mean percentage of Cauchy samples generated that had one or both sample moments fail to exist. To address this problem in the Cauchy case, the power study algorithm was modified to avoid counting samples which did not have sample moments. In other words, such samples were ignored for testing, and the total number of samples (40,000) was reduced by the amount that were discarded. The power study results for the Cauchy case, then, are valid.

3.5.3 Power Study Implementation. Once the structure of the power study and the types of alternate distributions were identified, the sequential test could be subjected to analysis. The

Table 3.7 Third and Fourth Moments of Alternate Distributions

Distribution	Skewness $\sqrt{\beta_1}$	Kurtosis β_2
Weibull $\beta = 0.5$	6.619	87.720
Weibull $\beta = 1$	2.000	9.000
Weibull $\beta = 1.5$	1.072	4.390
Weibull $\beta = 2$	0.631	3.245
Weibull $\beta = 2.5$	0.359	2.857
Weibull $\beta = 3$	0.168	2.730
Weibull $\beta = 3.5$	0.025	2.713
Weibull $\beta = 4$	-0.087	2.748
Beta (2,2)	0.000	2.143
Beta (2,3)	0.286	2.357
Gamma(2,1)	1.414	6.000
Normal (0,1)	0.000	3.000
Uniform (0,2)	0.000	1.800
$\chi^2(1)$	2.828	15.000
$\chi^2(4)$	1.414	6.000
Lognormal(0,1)	6.184	113.900
X Logistic	DNE	DNE
X Double Exponential	DNE	DNE
X Cauchy (0,1)	DNE	DNE

Table 3.8 Cauchy Samples With Non-Existent $\sqrt{b_1}$ or b_2

Cauchy Sample Size	20	30	50
Mean Number Observed	1268	1915.3	3141
Percentage	3.17%	4.79%	7.85%

first evaluation was conducted using the sequential test on both the Bush and Wozniak alternatives. As with the Bush cases, this study added the sample size 50 to the assessments against Wozniak's alternatives. Following those results, the next task was to analyze the two-sided kurtosis and skewness tests separately against the same set of alternatives to determine which tests were most powerful against particular distributions. This information is valuable to help determine the significance levels for the two tests when they are employed sequentially. If one of the two tests demonstrates significantly better power against specific alternatives, and one is trying to decide the appropriate combination of significance levels to choose, opting for a higher significance level for the more powerful of the two tests will result in better power for the procedure.

The final avenue of inquiry for the power study focused on using directional one-sided versions of the skewness and kurtosis tests individually and sequentially to quantify any expected increase

in power. Intuitively, a one-sided test should yield better power than a two sided version, but it presupposes the user has prior knowledge as to which tail to test, upper or lower. To use the upper tailed skewness test, for example, the user must have *a priori* information indicating his data has potentially greater skew than that of the Weibull shape hypothesized for the sample. An examination of histograms of the given sample may yield some of these insights, as would some theoretical knowledge concerning the system producing the data. For example, in a reliability context, one may be evaluating failure times under the assumption of a constant failure rate (Weibull shape $\beta = 1$). If there is concern that the data actually arises from a system with a *increasing* failure rate, thus indicating a Weibull model with $\beta > 1$, then the test could be configured in a format to specifically address this potentiality. In this case, since larger Weibull shapes lead to lower skewness and kurtosis values, one would use lower-tailed versions of one or both of the tests. Certainly, this will not commonly be the case, but documenting the improvement in power was worthy of investigation. Indeed, if such information is available, it may be advisable to simply use the more powerful of the two tests individually, since the power of the sequential test will usually not exceed that of its strongest component at the same significance level. The advantage to the sequential procedure, though, is that it hypothetically provides more consistent power across a broader range of alternatives than the separate tests might.

The choice of the appropriate tail to test for the alternatives examined was made by examining the theoretical values for the moments of the distributions as noted in Table 3.7, and comparing those to the particular Weibull null hypothesis being tested. For instance, for testing the Weibull shape $\beta = 1$ with $\sqrt{\beta_1} = 2$ and $\beta_2 = 9$ against a normal alternative with $\sqrt{\beta_1} = 0$ and $\beta_2 = 3$, one could use a lower-tail skewness test or a lower-tail kurtosis test. Power studies for the individual one-sided tests were only conducted on a small subset of the alternate distributions as listed in Tables 3.9 and 3.10. The use of the upper tail tests against the transformed Logistic in both cases was based upon observation of the considerably large values of the sample moments in the earlier Monte Carlo simulations. The impact of incorporating directional versions of the component tests

Table 3.9 One-Sided Skewness Tests Conducted

H_0	H_a	Tail Tested
Weibull(β=1)	Weibull(β= 3.5)	Lower
	Normal(0,1)	Lower
	Gamma(2,1)	Lower
	Lognormal	Upper
	X Logistic	Upper
Weibull($\beta = 3.5$)	Weibull($\beta = 1$)	Upper
	Weibull($\beta = 2$)	Upper
	Beta(2,3)	Upper

Table 3.10 One-Sided Kurtosis Tests Conducted

H_0	H_a	Tail Tested
Weibull($\beta = 1$)	Weibull($\beta = 3.5$)	Lower
	Normal(0,1)	Lower
	Uniform(0,2)	Lower
	Lognormal	Upper
	X Logistic	Upper
Weibull($\beta = 3.5$)	Beta(2,2)	Lower
	Uniform(0,2)	Lower
	Weibull($\beta = 1$)	Upper
	Weibull($\beta = 2$)	Upper

into the sequential procedure was also explored, but only for the lognormal alternative against the Weibull shape $\beta = 1$ null hypothesis. Table 3.11 summarizes the battery of power studies conducted in this research.

3.6 Conclusion

For the development and implementation of this new sequential goodness-of-fit test, the critical values for the test statistics for the particular Weibull shapes considered here were needed. An extensive set of Monte Carlo simulations were coded and run to generate accurate critical value tables for the upper and lower tails of both the $\sqrt{b_1}$ and b_2 test statistics at a granularity sufficient for the two-sided tests. That however, was not enough to implement the new test, because a sequential procedure introduces the need for determining the attained significance levels resulting from combining two separate tests at their own levels of significance. Again, large Monte Carlo simulations using a technique modified from Onen's work were the tool of choice. Once these and

the critical values had been empirically estimated in this fashion, the sequential test could finally be put to work against samples from various alternate distributions to examine its utility in terms of its power. Hence, numerous power studies were conducted against a host of alternate distributions for the sequential test, the individual skewness and kurtosis tests, and various one-sided variants of all three to document their performances and compare them to existing EDF test results. The results of these efforts provide the reader with the means to implement the new test and assess its effectiveness.

Table 3.11 Summary of Power Studies

			Tests Examined				
H_0	H_a	Sample Sizes	Sequential	Two-sided Skewness	Two-sided Kurtosis	One-sided Skewness	One-sided Kurtosis
$\beta = 0.5$	$\chi^2(1)$	20,30,50	X	X	X		
	X Logistic	"	X	X	X		
	X Double Exponential	"	X	X	X		
	X Cauchy	"	X	X	X		
$\beta = 1$	Beta(2,2)	5,15,25,50	X	X	X		
	Beta(2,3)	"	X	X	X		
	Gamma(2,1)	"	X	X	X	X	
	Normal(0,1)	"	X	X	X	X	X
	Uniform(0,2)	"	X	X	X		X
	Weibull β=1*	"	X	X	X		
	Weibull β=2	"	X	X	X		
	Weibull β=3.5	"	X	X	X	X	X
	$\chi^2(1)$	20,30,50	X	X	X		
	$\chi^2(4)$	"	X	X	X		
	Lognormal(0,1)†	"	X	X	X	X	X
	X Logistic	"	X	X	X	X	X
	X Double Exponential	"	X	X	X		
	X Cauchy	"	X	X	X		
$\beta = 1.5$	$\chi^2(4)$	20,30,50	X	X	X		
$\beta = 3.5$	Beta(2,2)	5,15,25,50	X	X	X		X
	Beta(2,3)	"	X	X	X	X	
	Gamma(2,1)	"	X	X	X		
	Normal(0,1)	"	X	X	X		
	Uniform(0,2)	"	X	X	X		X
	Weibull β=1	"	X	X	X	X	X
	Weibull β=2	"	X	X	X	X	X
	Weibull β=3.5*	"	X	X	X		

* Verification Runs

† One-sided variants of the sequential test also studied for this distribution

IV. RESULTS AND ANALYSIS

4.1 Introduction

The Monte Carlo simulations outlined in the previous chapter provide both the tools to implement the new sequential procedure for testing Weibull goodness-of-fit as well as measures of the test's effectiveness. The critical values and attained significance levels are fundamental components of the sequential test, without which the procedure is useless. Once the sequential test is developed, a determination of whether or not it contributes anything to the rich field of goodness-of-fit tests must be made by examining its power against specific alternate distributions. A careful assessment of the critical values, attained significance levels, and power-study results enable the analyst to make prudent decisions on how, when, and whether to utilize this new goodness-of-fit test.

4.2 Distribution of $\sqrt{b_1}$ and b_2

Before examining the actual critical values that were generated from the first set of simulations, it is helpful to explore the behavior of the joint distribution of sample skewness and sample kurtosis for the various Weibull shapes and sample sizes identified earlier. While D'Agostino and Stephens present appealing 90 and 95% contour plots on the $(\sqrt{b_1}, b_2)$ plane for the joint distribution of sample skewness and kurtosis for the Normal distribution [69: 282], no such work exists for Weibull samples; thus, a less sophisticated approach will be taken here. To simply visualize the general form of the distribution and the relationship between the two moments, the first 25,000 samples from the Monte Carlo simulation to generate critical values were plotted on the $(\sqrt{b_1}, b_2)$ plane and compared to the theoretical values found earlier and noted in Table 3.7. The plots for Weibull shapes 0.5,1,2, and 4 are presented in Figures 4.1 through 4.4. In each set, the theoretical values for the true moments are plotted with an '*' where the axis scaling permits. From these

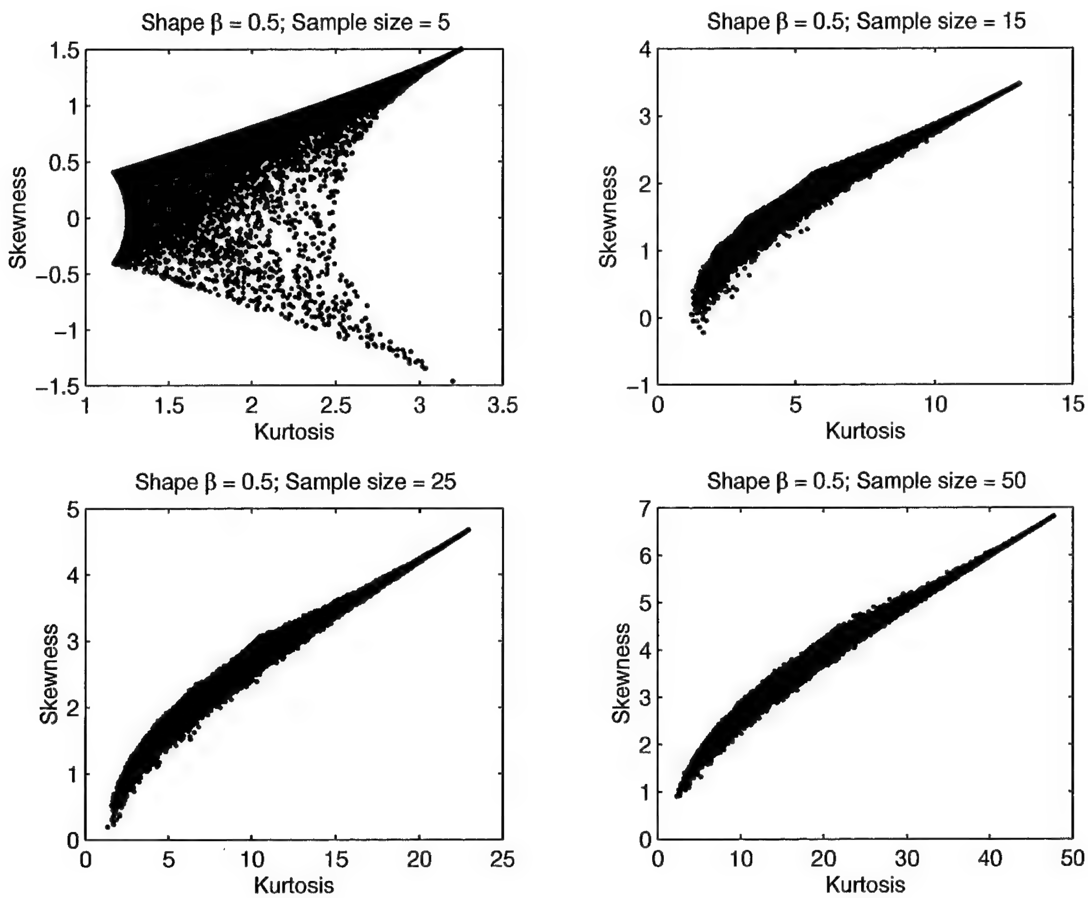


Figure 4.1 Distribution of $\sqrt{b_1}$ and b_2 for Shape $\beta = 0.5$.

plots one can identify several notable trends and surmise a few things about the behavior of these Weibull sample moments.

One of the more obvious characteristics of $\sqrt{b_1}$ and b_2 that emerges is that they tend to be highly correlated for smaller shapes and less so for large shapes. As sample skewness increases in magnitude so does sample kurtosis, although this trend becomes less apparent for shapes greater than 2.5 with sample sizes 25 and larger. This degree of correlation for smaller shape values indicates that one of the two moment-based tests may emerge as dominant in terms of power for these lower shapes because the test statistics are not measuring independent traits of the sample. The test statistic whose distribution has the least variability will probably demonstrate this dominance

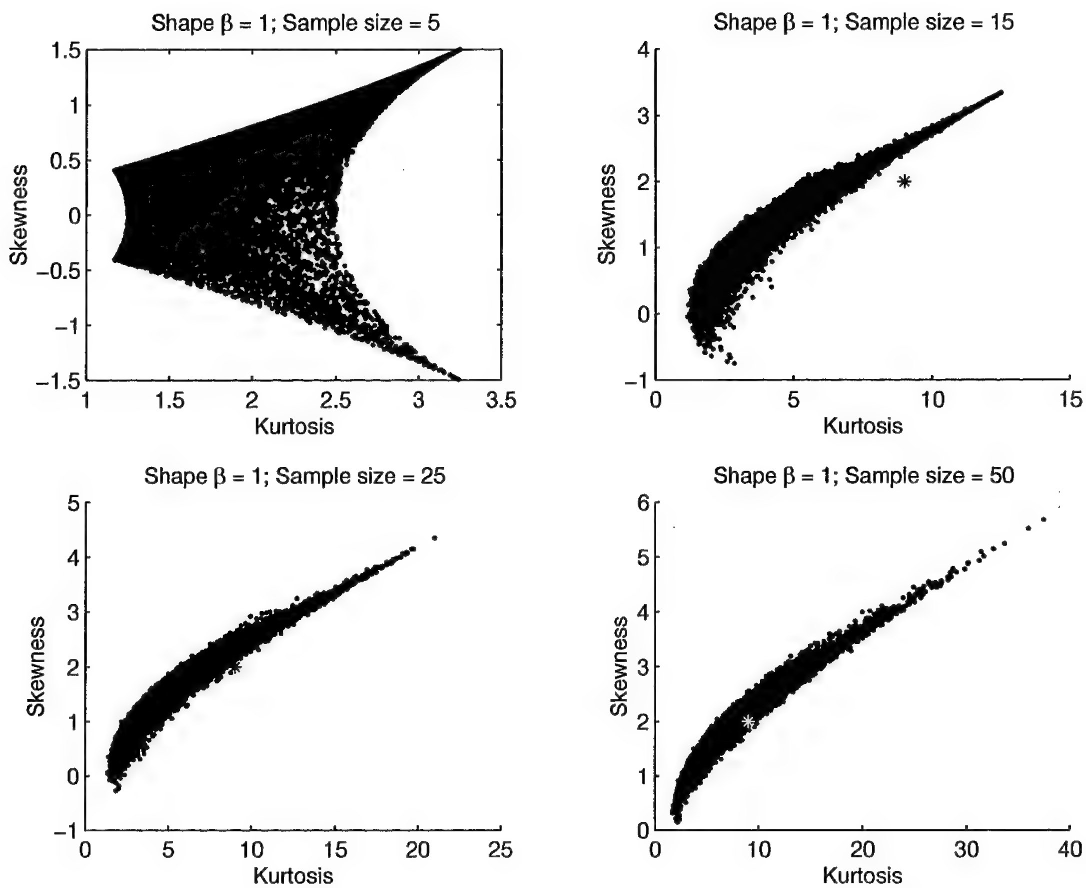


Figure 4.2 Distribution of $\sqrt{b_1}$ and b_2 for Shape $\beta = 1$.

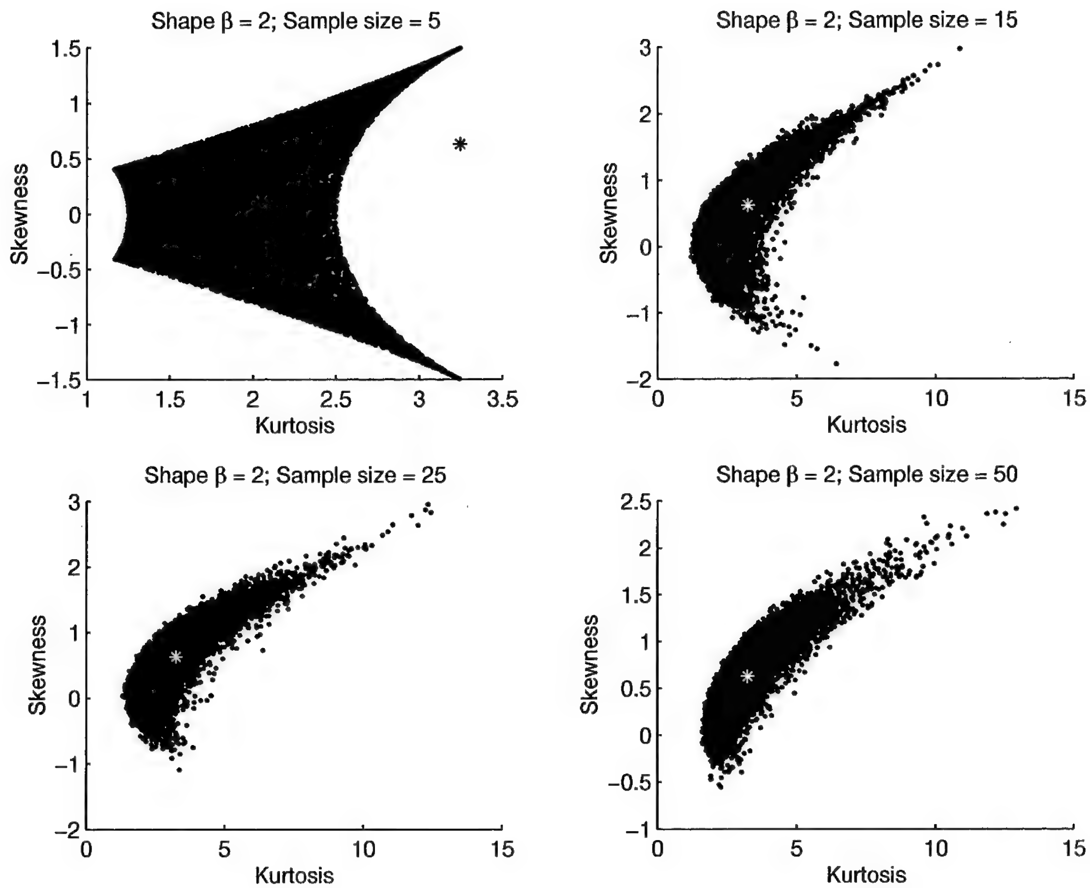


Figure 4.3 Distribution of $\sqrt{b_1}$ and b_2 for Shape $\beta = 2$.

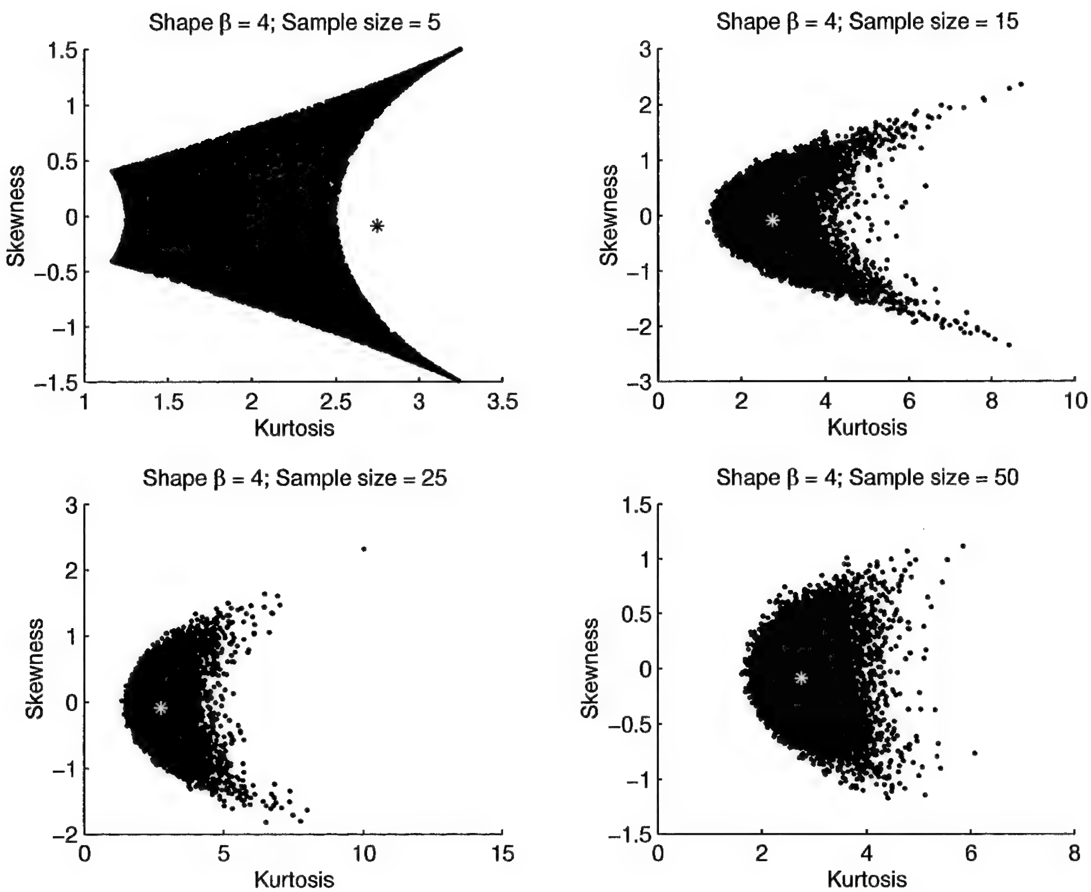


Figure 4.4 Distribution of $\sqrt{b_1}$ and b_2 for Shape $\beta = 4$.

since it tolerates less deviation from its central tendency. This effect should be lessened, however, as shape increases. The power study will substantiate this observation.

One also finds that as shape increases, the variability of $\sqrt{b_1}$ and b_2 change in relation to sample size. For shapes less than 2, both $\sqrt{b_1}$ and b_2 seem to assume a wider range of values as sample size increases. The trend stops at $\beta = 2$, and reverses as shape exceeds 2; variability in both sample moments diminishes as sample size increases. Intuitively, this behavior makes sense because at $\beta = 0.5$ or 1, the distribution is exponential or exponential-like, indicating a high degree of variance. It stands to reason that the sample moments would reflect this variability. Furthermore, at these shape values, the magnitudes of the true moments, $\sqrt{\beta_1}$ and β_2, are fairly large, so as sample size increases, the sample moments should also increase to approach the theoretical values. As shape increases to 2 and beyond though, the Weibull PDFs become more and more mound-shaped, which would lead to less variability in skewness and tail length. We also see that as shape increases, the joint distribution becomes more densely clustered about the theoretical values for the moments. What these findings indicate is that for tests with a known shape less than 2, it may be hard for these test statistics to distinguish between distributions because both $\sqrt{b_1}$ and b_2 assume such a wide range of values.

Yet another unmistakable observation is the unique nature of the distribution for sample size $n = 5$, regardless of shape. The scatterplot seems to have very fixed boundaries which are identical for all shape values, but the density is concentrated in the positively skewed portion for small shapes and becomes more uniform as shape increases. This interesting pattern must somehow be the product of the limits on the ranges of values generated by the expressions for $\sqrt{b_1}$ and b_2 given only five input quantities. What is important, however, is whether the pattern is substantially different for other probability distribution functions, meaning good discriminatory ability for the sequential test at very small sample sizes. To examine this question, 10,000 samples of size 5 were generated from each of the Beta(2,2), Lognormal(0,1), and Normal(0,1) distributions and plotted

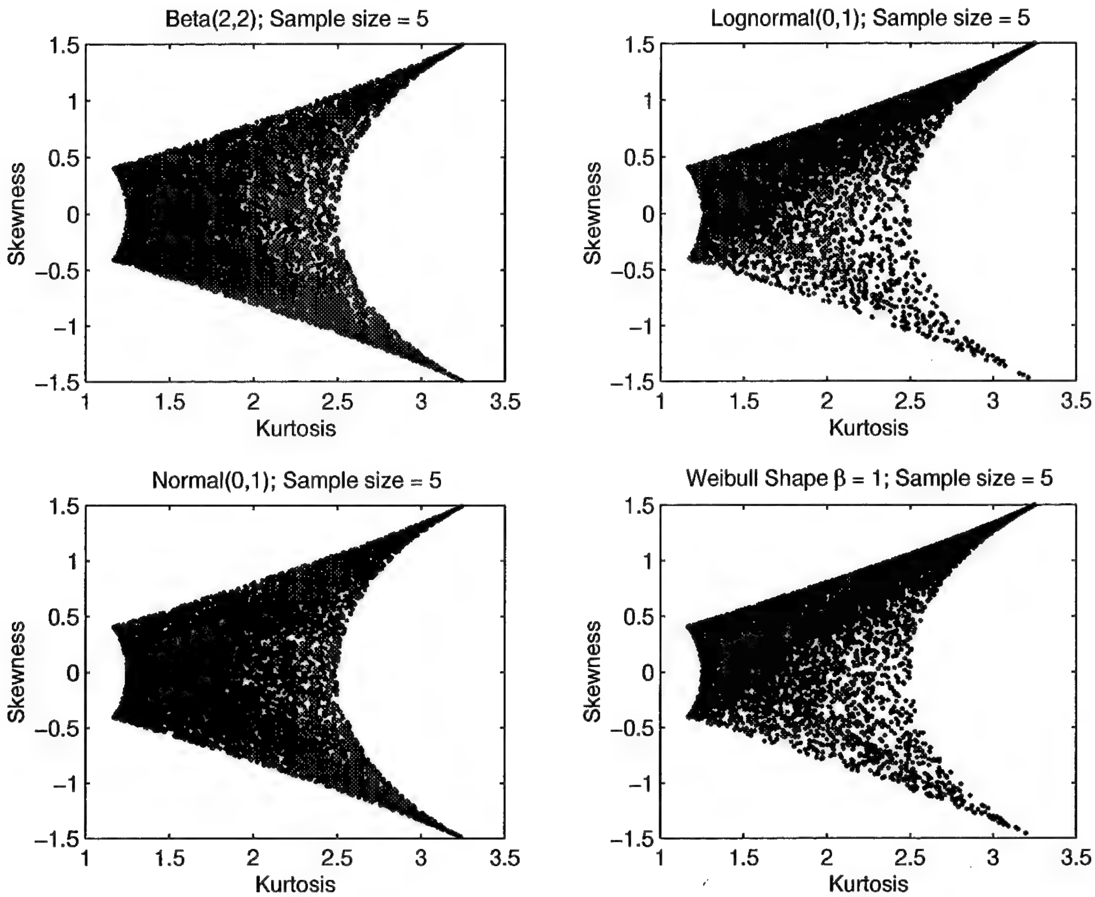


Figure 4.5 Distribution of $\sqrt{b_1}$ and b_2 for Various Distributions with Sample Size 5.

on the $(\sqrt{b_1}, b_2)$ plane. Figure 4.5 illustrates these plots compared to that of the Weibull(1,1). One finds identical boundaries, but differing patterns in the densities. It remains to be seen in the power study whether these differences are sufficient to translate to good power at small sample sizes.

4.3 Critical Values

4.3.1 The Critical Value Tables. Turning to more concrete simulation results, the critical value tables generated for this effort provide the upper- and lower-tail percentage points for $\sqrt{b_1}$ and b_2 at sample sizes $n = 5(5)50$ for each Weibull shape $\beta = 0.5(0.5)4$. Each table lists values for all significance levels $\alpha = 0.005(0.005)0.10$ and $0.10(0.01)0.20$. Again, this level of granularity was required by the two-sided nature of each test. The complete tables for each shape, sample size, and

Table 4.1 Kurtosis (b_2) Lower/Upper Tail Critical Values (Partial): $\beta = 1.0$

Sample	Significance Level (α)									
Size	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
5	1.185	1.196	1.207	1.216	1.224	1.232	1.239	1.246	1.252	1.257
10	1.296	1.351	1.386	1.419	1.445	1.469	1.491	1.514	1.533	1.551
15	1.447	1.515	1.564	1.600	1.634	1.665	1.692	1.718	1.741	1.764
20	1.590	1.664	1.721	1.766	1.804	1.840	1.872	1.900	1.927	1.954
25	1.712	1.793	1.855	1.904	1.950	1.988	2.027	2.063	2.097	2.131
30	1.808	1.910	1.980	2.039	2.095	2.143	2.184	2.222	2.255	2.290
35	1.921	2.037	2.116	2.181	2.237	2.287	2.333	2.373	2.411	2.448
40	2.014	2.133	2.222	2.286	2.340	2.393	2.441	2.486	2.528	2.567
45	2.114	2.237	2.325	2.402	2.464	2.517	2.568	2.615	2.662	2.705
50	2.204	2.331	2.424	2.499	2.560	2.618	2.669	2.715	2.759	2.798

Sample	Significance Level (1-α)									
Size	0.995	0.99	0.985	0.98	0.975	0.97	0.965	0.96	0.955	0.95
5	3.231	3.220	3.208	3.196	3.184	3.172	3.160	3.149	3.137	3.125
10	7.473	7.242	7.062	6.917	6.783	6.668	6.556	6.448	6.355	6.250
15	10.876	10.280	9.813	9.470	9.178	8.941	8.708	8.494	8.296	8.114
20	13.601	12.610	11.886	11.395	10.958	10.584	10.251	9.974	9.724	9.463
25	15.524	14.218	13.377	12.751	12.229	11.764	11.383	11.009	10.681	10.353
30	17.572	15.831	14.737	13.879	13.224	12.670	12.168	11.785	11.425	11.094
35	18.674	16.670	15.524	14.657	13.931	13.300	12.800	12.320	11.940	11.574
40	19.964	17.869	16.510	15.500	14.674	14.024	13.487	12.965	12.555	12.180
45	21.008	18.393	16.865	15.827	15.050	14.354	13.806	13.314	12.878	12.517
50	21.684	19.038	17.430	16.294	15.457	14.692	14.063	13.565	13.117	12.679

significance level are provided in the appendix, but for illustrative purposes, a portion of the tables for sample kurtosis for $\beta = 1$ is given in Table 4.1.

To demonstrate use of the critical value tables, assume a sample of size $n = 15$ has been drawn, with $b_2 = 8.8$, from an hypothesized Weibull with shape $\beta = 1$. To conduct the kurtosis test at an $\alpha = 0.01$ significance level, one would look in the $\frac{\alpha}{2} = .005$ column of the lower tail table to find $b_{2(0.005)} = 1.447$ and in the $(1 - \frac{\alpha}{2}) = 0.995$ column of the upper tail table to identify $b_{2(0.995)} = 10.876$. These two values define the rejection region for the sample kurtosis goodness-of-fit test, and since $1.447 < b_2 = 8.8 < 10.876$, the conclusion would be to fail to reject the null hypothesis. Note, however, that if a different significance level were used, say $\alpha = 0.07$, the sample would fail the upper tail portion of the test since $b_{2(0.965)} = 8.708$. The sample skewness test would have to be conducted similarly to complete the sequential goodness-of-fit procedure. Further examples will follow the presentation of the results for the attained significance levels.

4.3.2 Estimates of Variability. Since the critical values presented in this paper are the products of Monte Carlo simulation, and are, thus, estimates whose accuracy is driven by the number of replications performed, it is appropriate to examine the degree of variability present in them to establish confidence in their use. To quantify this, several additional simulations were run, and the standard deviations of the corresponding resultant critical values were measured. Given the long run-times for single 100,000-trial runs, only three such runs were made at shape values $\beta = 0.5$ and 1 to provide at least 3 samples of each critical value from which a rough estimate of variance could be calculated. Only these two smaller shapes were evaluated because they seemed to demonstrate the greatest variability in sample skewness and kurtosis as evidenced by the scatter plots discussed above and by the wide range in values spanned from lower tail to upper tail critical values in the tables. Thus, the estimates of standard deviations at $\beta = 0.5, 1$ would presumably serve as an upper bound on the estimates for the larger shapes, which seemed to exhibit less variability in the sample moments. The estimates found are shown in Tables 4.2 through 4.5.

Even though the smaller shape values exhibit more variability than the others, the standard deviations shown in the tables are quite acceptable — most are only significant in the third decimal place. Hence, one can have a fair amount of confidence in the critical values tabled from this study. To reduce the significance of the error to the fourth decimal place would require over 1 million trials at each sample size, according to Shooman's $\frac{1}{\sqrt{N}}$ rule noted earlier in Chapter 3 [63: 259]. Given the relatively low variance indicated above, this additional cost in computer times was deemed unacceptable. Another note to make here is that the standard deviations generally increase with sample size, consistent with the observations from the scatter plots for shapes less than 2 described above. Furthermore, one finds greater variability in the kurtosis values than skewness, consistent with the more variable nature of this sample moment at small shape values.

4.3.3 Trends in the Critical Values. Some useful insights can also be gleaned by examining the critical value tables closely and observing some general trends in their behavior according to

Table 4.2 Standard Deviations for Sample Skewness: $\beta = 0.5$

Sample Size	Significance Level					
	0.01	0.05	0.10	0.90	0.95	0.99
5	0.004	0.006	0.005	0.001	0.000	0.000
10	0.011	0.003	0.001	0.001	0.001	0.001
15	0.007	0.006	0.002	0.004	0.003	0.002
20	0.003	0.004	0.002	0.004	0.003	0.002
25	0.008	0.002	0.002	0.004	0.003	0.006
30	0.003	0.001	0.001	0.011	0.008	0.003
35	0.011	0.006	0.009	0.013	0.014	0.009
40	0.006	0.005	0.002	0.006	0.003	0.004
45	0.019	0.011	0.010	0.011	0.006	0.012
50	0.012	0.007	0.003	0.009	0.014	0.006

Table 4.3 Standard Deviations for Sample Skewness: $\beta = 1$

Sample Size	Significance Level					
	0.01	0.05	0.10	0.90	0.95	0.99
5	0.000	0.004	0.004	0.001	0.002	0.001
10	0.003	0.003	0.001	0.007	0.001	0.003
15	0.004	0.007	0.003	0.005	0.008	0.006
20	0.006	0.004	0.001	0.003	0.001	0.009
25	0.003	0.003	0.002	0.005	0.008	0.019
30	0.005	0.002	0.002	0.003	0.004	0.009
35	0.007	0.006	0.005	0.003	0.004	0.017
40	0.000	0.001	0.001	0.007	0.010	0.006
45	0.009	0.005	0.004	0.004	0.002	0.022
50	0.007	0.003	0.002	0.003	0.004	0.006

shape and sample size and comparing them to the findings from the scatter plots. One means to visualize these trends is to plot the upper and lower tail critical values at a couple significance levels against sample size for a given Weibull shape. The plots for $\sqrt{b_1}$ and b_2 for $\beta = 0.5, 1, 2, 3,$ and 4 are shown in Figures 4.6 to 4.10. It is helpful to keep in mind the true theoretical values for $\sqrt{\beta_1}$ and β_2 from Table 3.7 while examining the plots. In each plot the α-levels plotted are the 0.01, 0.10, and 0.20 levels in both the upper and lower tail. The lower tail values are joined with a solid line while the upper tail is in dot-dash format.

The plots yield some valuable observations on the behavior of the test statistics. For all values of shape, the lower tail critical values for both $\sqrt{b_1}$ and b_2 are monotonically increasing with sample size. The behavior of the upper tail, however, varies by shape. For $\beta \leq 1$, the upper tail values are monotonically increasing, but at $\beta = 1.5$, the upper tail $\sqrt{b_1}$ magnitudes remain relatively constant, while the b_2 values continue to increase, but not as fast as with smaller shapes. Starting at $\beta = 2$

Table 4.4 Standard Deviations for Sample Kurtosis: $\beta = 0.5$

Sample Size	Significance Level 0.01	0.05	0.10	0.90	0.95	0.99
5	0.001	0.001	0.004	0.001	0.000	0.000
10	0.003	0.000	0.005	0.004	0.005	0.002
15	0.011	0.008	0.008	0.021	0.010	0.009
20	0.006	0.007	0.003	0.020	0.017	0.011
25	0.008	0.004	0.009	0.020	0.021	0.036
30	0.005	0.005	0.003	0.082	0.061	0.018
35	0.038	0.025	0.031	0.103	0.108	0.072
40	0.017	0.025	0.018	0.046	0.036	0.039
45	0.043	0.043	0.037	0.107	0.047	0.100
50	0.045	0.026	0.013	0.059	0.135	0.054

Table 4.5 Standard Deviations for Sample Kurtosis: $\beta = 1$

Sample Size	Significance Level 0.01	0.05	0.10	0.90	0.95	0.99
5	0.000	0.001	0.002	0.001	0.002	0.002
10	0.000	0.001	0.001	0.016	0.005	0.007
15	0.007	0.002	0.002	0.022	0.042	0.016
20	0.007	0.005	0.004	0.005	0.016	0.039
25	0.004	0.005	0.003	0.025	0.052	0.127
30	0.003	0.003	0.001	0.041	0.023	0.064
35	0.009	0.008	0.011	0.020	0.043	0.152
40	0.004	0.002	0.003	0.039	0.060	0.019
45	0.012	0.016	0.008	0.037	0.037	0.196
50	0.011	0.007	0.005	0.019	0.048	0.086

and continuing throughout the higher shapes, both upper tails begin to demonstrate a decreasing tendency with increasing sample size. The $\sqrt{b_1}$ lower tail exhibits this tendency more clearly than b_2. In fact, one notes that for all $\beta > 2$, the upper tail b_2 values actually increase initially from sample size 5 to 10 or 15, before beginning a decreasing trend.

It is known that as shape increases the true skewness and kurtosis values decrease as seen in Table 3.7. One would expect that as sample size increases, the upper and lower tail critical values would tend to converge about these theoretical values. Indeed this is what one finds in the plots. By examining the trend lines, it is clear that sample skewness critical values converge more rapidly than those for kurtosis. The high variability in b_2 seen in the scatter plots is also apparent in these graphs. In all cases, the critical values at the $\alpha = 0.01$ level in the upper tail show a significant spread from those at the $\alpha = 0.10$ and 0.20 levels, and the slope of convergence toward the theoretical values with increasing sample size is very slight. Note, however, that the lower

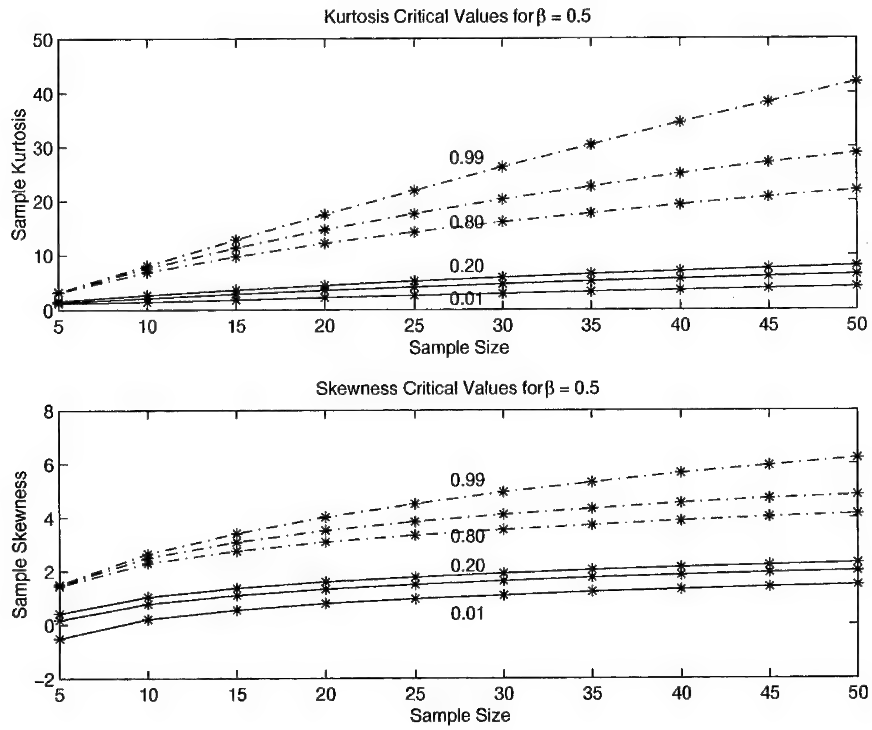


Figure 4.6 Critical Values for $\sqrt{b_1}$ and b_2 with Shape $\beta = 0.5$.

tail kurtosis values do not demonstrate this same variability. These observations suggest several things: first, the lower variability in $\sqrt{b_1}$ might make it a more powerful test statistic than b_2 in many cases; second, the kurtosis test will probably demonstrate greater discriminatory power when testing alternatives that have lower kurtosis values. Additionally, the jump in upper tail kurtosis critical values from $n = 5$ to 10 and the fact that all $n = 5$ critical values are nearly identical from shape to shape suggest that the distribution of b_2 is tightly constrained within particular limits. This is consistent with the unusual shape of the boundaries in the scatter plots for sample size 5 on the $(\sqrt{b_1}, b_2)$ plane, and may suggest good power for small sample sizes. It remains for the power study to empirically validate these indications.

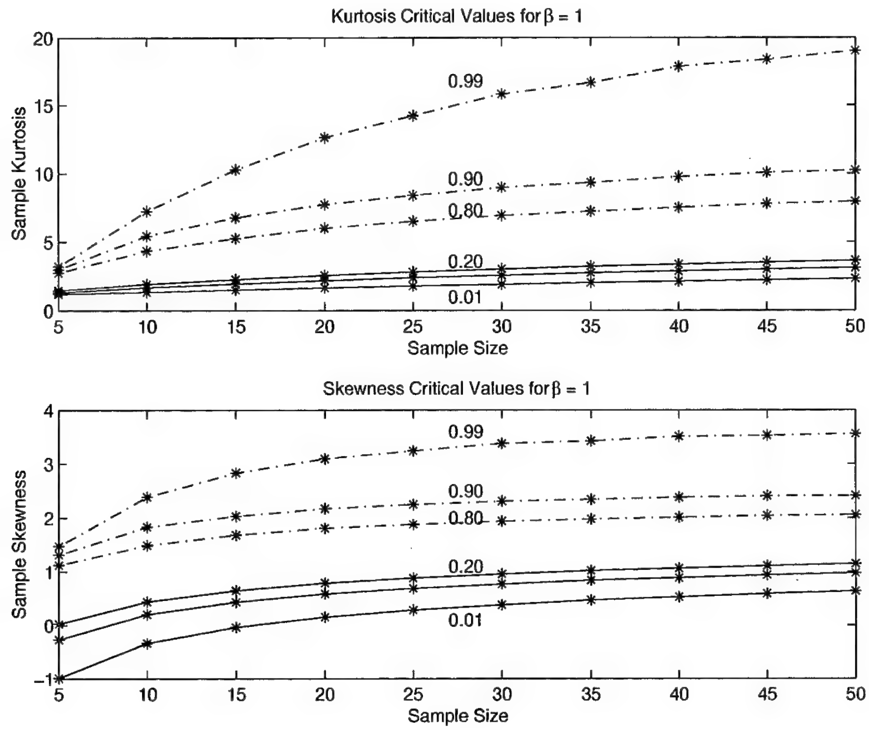


Figure 4.7 Critical Values for $\sqrt{b_1}$ and b_2 with Shape $\beta = 1$.

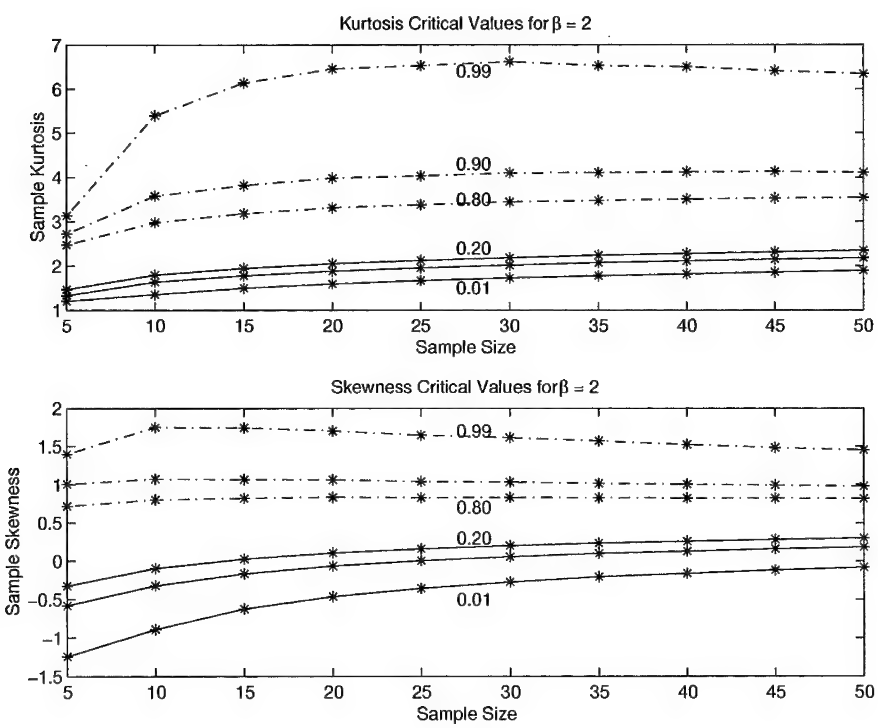


Figure 4.8 Critical Values for $\sqrt{b_1}$ and b_2 with Shape $\beta = 2$.

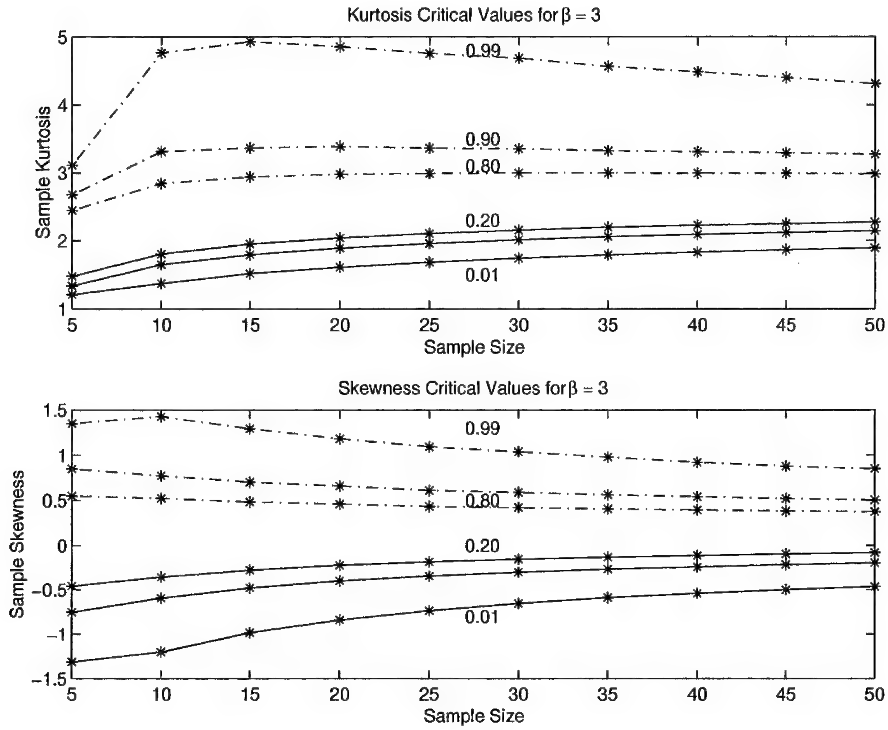


Figure 4.9 Critical Values for $\sqrt{b_1}$ and b_2 with Shape $\beta = 3$.

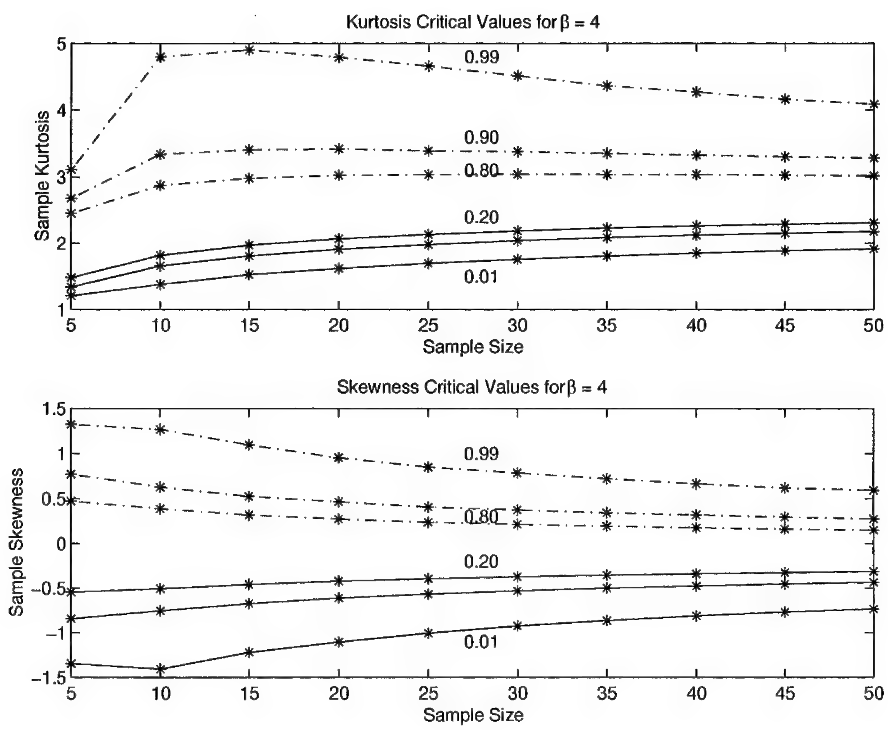


Figure 4.10 Critical Values for $\sqrt{b_1}$ and b_2 with Shape $\beta = 4$.

4.4 Attained Significance Levels

4.4.1 Tables vs. Contour Plots. Once the critical values for the two test statistics have been derived and analyzed, the next step in the development of the sequential test was determining the attained significance levels for the combined test. As discussed in Chapter 3, the individual tests can be paired at any number of significance levels, each of which yields a different overall significance level for the omnibus test. Without some assessment of these attained significance levels, the test would be useless because one needs to select the appropriate levels of the component tests such that when combined sequentially, they yield the desired overall significance. The Monte Carlo simulation described earlier to approximate these levels was conducted for all $\beta = 0.5(0.5)4$, sample sizes $n = 5(5)50$, and all combinations of significance levels for both tests from $\alpha = 0.01(0.01)0.20$.

The results of the calculations can be presented in tabular format as shown on the following page for $\beta = 1$ and sample size 10 (Table 4.6). To even the casual observer, it is clear that the use of such extensive tables can be cumbersome in actual practice. One has to locate in the table a significance level that equals or is nearly equal to the desired significance level for the test, then chose from the column and row headings the corresponding levels for the two component tests. Given this level of difficulty, a more useful presentation of the results was developed in the form of contour plots that depict the data in the tables. Tables and the corresponding contour plots for each shape/sample size combination have been produced, but only the more useful contour plots are reproduced in the appendix. The tables for $\beta = 1$ are also included for cross-reference purposes, and the other tables can be obtained on request from the author. The contour plot corresponding to the sample table on the following page is shown in Figure 4.11. This visual approach should significantly simplify the employment of the sequential test since one only needs to identify the contour line corresponding to the desired significance level and then select the appropriate combination of levels for the two tests by referring to the axes. The only real challenge is deciding upon which combination to use.

Table 4.6 Attained Significance Levels: Sample size = 10 ; $\beta = 1$

		Kurtosis Test Significance Level (α)																			
		0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20
Skewness Test Significance Level (α)	0.01	0.015	0.025	0.035	0.044	0.054	0.063	0.072	0.082	0.092	0.102	0.112	0.122	0.131	0.142	0.153	0.163	0.173	0.183	0.193	0.202
	0.02	0.025	0.030	0.039	0.049	0.058	0.068	0.077	0.087	0.096	0.107	0.116	0.126	0.135	0.146	0.157	0.167	0.177	0.187	0.196	0.206
	0.03	0.035	0.040	0.045	0.053	0.063	0.072	0.081	0.091	0.100	0.111	0.120	0.130	0.139	0.150	0.160	0.170	0.180	0.190	0.200	0.209
	0.04	0.044	0.049	0.053	0.058	0.067	0.076	0.085	0.095	0.104	0.114	0.124	0.133	0.142	0.153	0.163	0.173	0.183	0.193	0.202	0.212
	0.05	0.054	0.058	0.062	0.067	0.072	0.080	0.089	0.099	0.108	0.118	0.127	0.137	0.146	0.156	0.166	0.176	0.186	0.195	0.205	0.214
	0.06	0.063	0.068	0.071	0.076	0.080	0.086	0.093	0.103	0.112	0.122	0.131	0.140	0.149	0.159	0.169	0.179	0.189	0.198	0.208	0.217
	0.07	0.073	0.077	0.081	0.085	0.089	0.093	0.098	0.106	0.115	0.125	0.134	0.143	0.152	0.162	0.172	0.181	0.191	0.200	0.210	0.219
	0.08	0.083	0.086	0.090	0.094	0.098	0.102	0.105	0.111	0.119	0.128	0.137	0.146	0.155	0.165	0.175	0.184	0.194	0.203	0.212	0.221
	0.09	0.091	0.095	0.098	0.102	0.106	0.110	0.113	0.118	0.124	0.131	0.140	0.149	0.157	0.167	0.177	0.186	0.196	0.205	0.214	0.223
	0.10	0.101	0.105	0.108	0.111	0.115	0.119	0.122	0.126	0.131	0.137	0.144	0.152	0.160	0.170	0.180	0.189	0.198	0.207	0.216	0.225
	0.11	0.111	0.114	0.117	0.120	0.124	0.127	0.131	0.135	0.139	0.143	0.149	0.156	0.163	0.173	0.182	0.191	0.201	0.210	0.219	0.228
	0.12	0.120	0.123	0.126	0.130	0.133	0.136	0.139	0.143	0.147	0.151	0.155	0.161	0.167	0.176	0.185	0.194	0.203	0.212	0.221	0.230
	0.13	0.129	0.132	0.135	0.138	0.141	0.144	0.147	0.151	0.155	0.159	0.163	0.167	0.172	0.180	0.188	0.197	0.206	0.214	0.223	0.232
	0.14	0.140	0.142	0.145	0.148	0.151	0.154	0.157	0.160	0.164	0.168	0.171	0.175	0.179	0.185	0.192	0.200	0.209	0.217	0.226	0.234
	0.15	0.149	0.151	0.154	0.157	0.160	0.162	0.165	0.169	0.172	0.175	0.179	0.182	0.186	0.191	0.197	0.204	0.212	0.220	0.228	0.237
	0.16	0.159	0.161	0.163	0.166	0.168	0.171	0.174	0.177	0.180	0.184	0.187	0.190	0.193	0.198	0.203	0.208	0.216	0.223	0.231	0.239
	0.17	0.169	0.171	0.173	0.176	0.178	0.181	0.183	0.186	0.189	0.192	0.196	0.198	0.202	0.206	0.210	0.214	0.221	0.227	0.235	0.242
	0.18	0.179	0.181	0.183	0.185	0.187	0.190	0.192	0.195	0.198	0.201	0.204	0.207	0.210	0.214	0.217	0.221	0.227	0.232	0.239	0.246
	0.19	0.188	0.190	0.192	0.194	0.196	0.199	0.201	0.204	0.206	0.209	0.212	0.215	0.218	0.222	0.225	0.229	0.233	0.238	0.244	0.250
	0.20	0.198	0.199	0.201	0.203	0.205	0.208	0.210	0.212	0.215	0.218	0.221	0.223	0.226	0.230	0.233	0.236	0.240	0.244	0.249	0.255

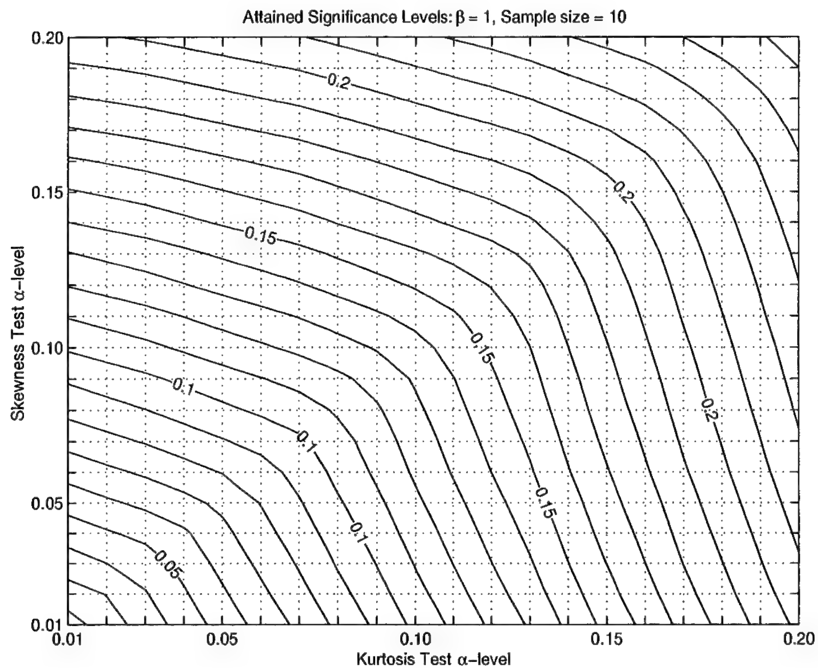


Figure 4.11 Contour Plot for Attained Significance Levels: $\beta = 1$, Sample Size $= 10$.

4.4.2 Estimates of Variability. To approximate the variance in these estimates for the attained significance levels, a procedure identical to that noted earlier for estimating the variability in the power study results was followed. Since these significance levels are proportions, the expression (3.4) serves as the estimate for standard error. The N in the expression is 100,000 — the number of trials of each sample size that were run to generate these results. Since it is p that is being estimated, one cannot evaluate the expression directly, but it is possible to determine an upper bound using $p = 0.5$ because the expression is maximized there. In reality, this estimate will be an extremely conservative one since the largest significance level observed in the tables rarely exceeds 0.30. Thus, with $N = 100,000$ and $p = 0.5$, one finds that $\sigma_{\widehat{p}} = 0.00158$. Since this is so small, one can be confident in at least the first two decimal places of these attained significance level results.

4.4.3 Example of Use. An illustration of the use of these attained significance tables/plots is perhaps useful at this point. To employ the sequential test, the first decision will be the choice of

Table 4.7 Example: Potential Significance Level Combinations for Sequential Test.

Significance Levels		
$\sqrt{b_1}$ Test	b_2 Test	Sequential Test
0.01	0.10	0.102
0.03	0.09	0.100
0.07	0.07	0.098
0.09	0.04	0.102

the desired significance level for the overall test. Given that value, one uses the tables or contour plots to determine the significance levels of the individual skewness and kurtosis tests. Once this choice is made, the critical value tables are consulted to identify the upper and lower tail boundaries of the rejection region for the test, and the test can be evaluated.

For instance, suppose an analyst wants to conduct the sequential test at a level $\alpha = 0.10$ for a sample of size 10 with an hypothesized Weibull shape $\beta = 1$. By consulting the same table on page 4-16, he scans for values in the table close to 0.10. It is readily apparent that there are numerous possibilities, only one of which is 0.10. Several of these options are shown in Table 4.7.

If he has no prior knowledge about possible alternates to the null hypothesis, he may opt for nearly equal levels for each test; consequently, the third choice would be appropriate — conduct the skewness test at $\alpha_1 = 0.07$ and the kurtosis test similarly at $\alpha_2 = 0.07$. The corresponding critical values can be found in the tables in Appendix A. For the skewness test at $\alpha_1 = 0.07$, the lower tail critical value is found in the $\frac{\alpha_1}{2} = 0.035$ column, and is -0.061; the upper tail is located in the $(1 - \frac{\alpha_1}{2}) = 0.965$ column, and is 2.176. For the kurtosis test, the lower tail value similarly is 1.491 and the upper tail is 6.556.

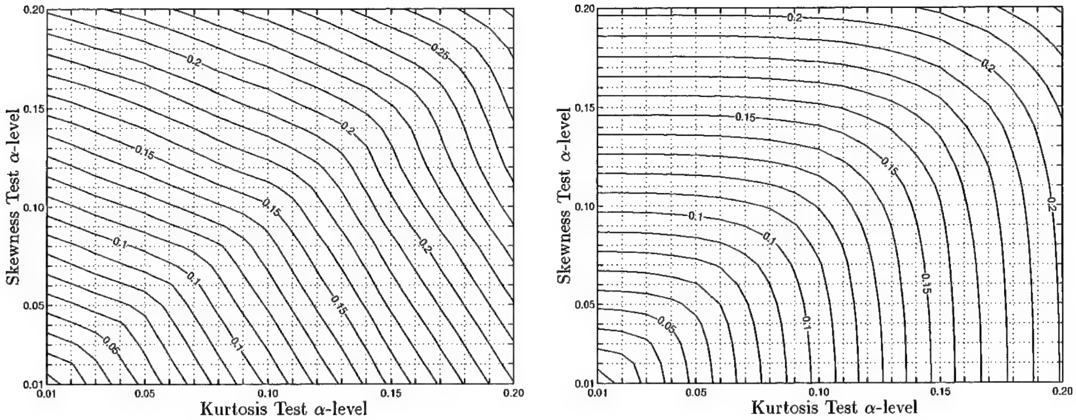
Now suppose that the test statistics for the observed sample are $\sqrt{b_1} = 1.245$ and $b_2 = 5.427$. Since $-0.061 < 1.245 < 2.176$ and $1.491 < 5.427 < 6.556$, neither test statistic falls in the rejection region prescribed by the critical values, and the analyst concludes that there is no grounds to reject the null hypothesis at the $\alpha = 0.10$ level. To increase his confidence or credibility, he may wish to evaluate the result for several of the combinations of α-values in Table 4.7 above.

Using the contour plots for the test is somewhat easier because one does not have to get lost in the sea of numbers that the tables present. To use the contour plot in Figure 4.11, the analyst needs only to find the 0.10 contour line and move along it until he finds a suitable combination of levels for the two tests. Here the $\alpha_1 = 0.07$ and $\alpha_2 = 0.07$ combination emerges quite readily.

A general rule to keep in mind when determining the appropriate levels of the two tests if one has some *a priori* information about potential alternatives is to pick the higher significance level for the test which he expects to be more powerful in discriminating among the possible alternatives. Recall that the larger the α-value, the larger the rejection region, and, hence, the greater the power. The power study that follows will yield insights into how each test performs against particular alternatives that will be useful in making these significance level choices. For situations where there is no information available on possible alternatives, it may be safest to pick the largest nearly equivalent α-levels for both tests. Realize, of course, that this choice will in all likelihood lead to slightly lower power, but it cannot be avoided if no information exists to justify a different selection.

Continuing with the previous example, if the analyst were more concerned about the sample coming from a skewed alternative rather than one differing in kurtosis, he would probably seek a combination in which the α-level for the skewness test was higher, such as 0.09. This would lead to higher Type I error for this skewness test, and, thus, higher power in rejecting skewed alternatives. In this case, the corresponding significance level for the kurtosis test would be 0.04 as shown in Table 4.7.

4.4.4 Observations on the Significance Levels. An examination of the contour plots of the significance levels leads to some valuable observations that may contribute to effectively utilizing the sequential test. As with the critical values, there is a clear distinction between behavior for Weibull shapes less than 2 and those greater. For $\beta \in \{0.5, 1, 1.5\}$ one finds that as sample size increases, the contour levels demonstrate increasing curvature with larger sample sizes to the point



(a) Sample Size 5 (b) Sample Size 50

Figure 4.12 Significance Levels for $\beta = 1$.

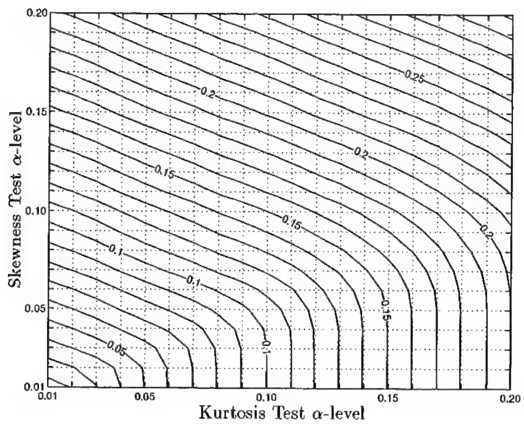
they nearly align with the matching significance levels of the individual tests. By sample size 50, for instance, with $\beta = 1$ shown in Figure 4.12, the 0.10 contour level lines up almost directly with the x-y grid lines for $\alpha = 0.10$ for both skewness and kurtosis tests. Compare this with the adjacent plot for sample size 5. The difference is clear. What this indicates is that for large sample sizes, both tests can be conducted at high significance levels close to that of the overall test, possibly leading to better power performance. Referring to the $n = 50$ plot, one could conduct both the skewness and kurtosis tests at the 0.08 level and maintain an overall level of 0.10, rather than some more biased combination like $\alpha_1 = 0.06$ and $\alpha_2 = 0.09$. Since power increases with increasing α, one would like the α-levels of both the component tests to be as close as possible to that of the sequential test to preserve power, particularly in cases where no prior information on alternates would suggest which of the two tests may be more powerful.

To see this, understand that at a given significance level, a sequential test typically has lower power against a particular alternative than the most powerful test it utilizes due to the effect of Bonferroni's inequality. For a sequential test at $\alpha = 0.05$, for example, each individual test usually must be evaluated at some combination of smaller significance levels, say 0.04 and 0.02, where they have less power than they would exhibit individually at the 0.05 level. Hence, the sequential

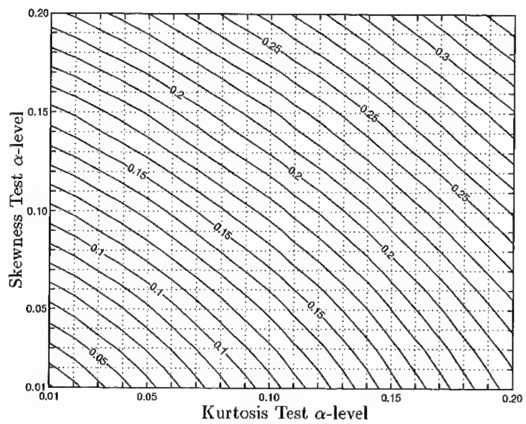
test does not always "catch up" to the power of the best of its components when compared at the same significance level. (The advantage, of course, is that it may demonstrate better average power against a broad range of alternatives, as each component test excels against particular alternatives. In some cases, the tests will complement each other and produce better power in sequential form than as stand-alone tests. Each of these situations can be seen in the power study.) However, in the situation observed here for $\beta < 2$ and large sample sizes, one can run both component tests at significance levels close to that of the overall test, minimizing this lag between power of the sequential test and the component tests. This is a beneficial feature of the significance levels for small shapes and large sample sizes.

This is not the case for $\beta \geq 2$. In these cases, there is no trend toward increasing curvature of the contour levels as sample size increases, as seen in Figure 4.13 for $\beta = 3$ for sample sizes 5 and 50. Here, the contours actually become more linear with larger samples. What this means is that for shapes greater than 2, it is more advantageous to pick a combination of α-levels for the component tests which favors the more powerful test. To illustrate, suppose one resorts to choosing nearly equivalent levels for a test at $\alpha = 0.10$ and sample size 50 for this shape. One possible combination would be $\alpha_1 = 0.06$ and $\alpha_2 = 0.05$, both of which are substantially lower than the overall level of 0.10. If one was testing against skewed alternatives, he should opt for a higher skewness significance level, resulting in a combination such as $\alpha_1 = 0.08$ and $\alpha_2 = 0.02$ or $\alpha_1 = 0.09$ and $\alpha_2 = 0.01$. Of course, if there is no information that may indicate which test is stronger against a given alternative, one may have to resort to using nearly equivalent significance levels and thereby expect lower power.

Another trend manifest in the contour plots for $\beta \geq 2$ is the increasing density of the contours along the diagonal of equal significance levels ($\alpha_1 = \alpha_2$). Comparing the plot for $\beta = 2$, $n = 25$ to that of $\beta = 4$, $n = 25$ in Figure 4.14, one notices that while specific contour levels are anchored along the axes in nearly identical locations, there are more contour lines in the upper right corner,



(a) Sample Size 5

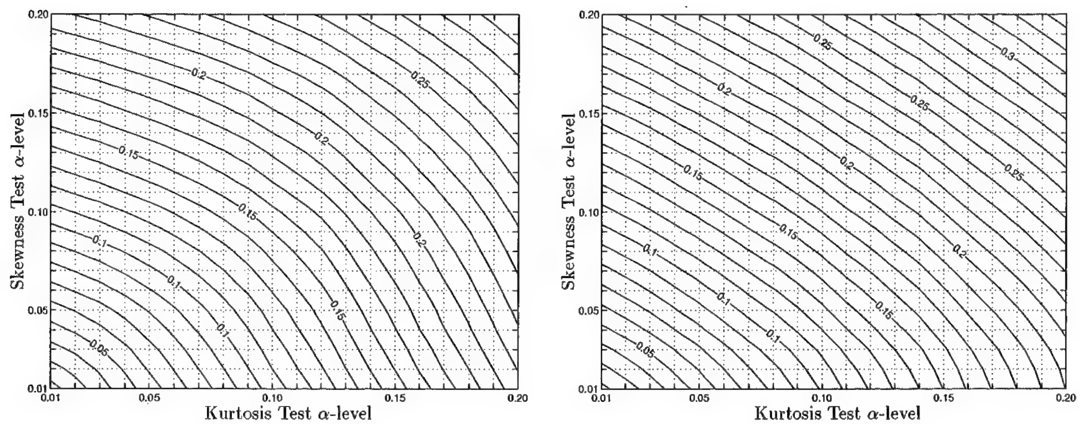


(b) Sample Size 50

Figure 4.13 Significance Levels for $\beta = 3$.

meaning that at a given combination of equivalent contour levels (e.g. $\alpha_1 = \alpha_2 = 0.15$) the attained significance level is higher for higher shape values. The fact combined with the earlier observations for $\beta < 2$ indicates that for such equal or nearly equal significance levels, one can expect more power with smaller shapes because the lag between significance levels of the sequential test and its components is substantially smaller for the lower shape values.

This discussion leads to some basic tactics for picking the significance levels of the component tests based on the form of the contour plots that could help optimize power. If the contour levels demonstrate a high degree of curvature (as with $\beta < 2$ and n large), one can pick nearly equal levels for both skewness and kurtosis tests and not have to compromise power for cases when little or no information is available on the alternatives. Now, if some such data is available, using a higher significance level for the more powerful test is still prudent, but gains in power probably will not be outstanding. On the other hand, for contour plots with more linear contour levels (as with $\beta > 2$), using nearly equal significance levels is clearly not the best option in terms of power, especially as shape increases. There may not be a choice, but if it is possible to identify which test may be more powerful against the alternatives being evaluated, using the higher significance level



(a) $\beta = 2$ (b) $\beta = 4$

Figure 4.14 Significance Levels for $\beta = 2$ vs. $\beta = 4$, Sample size 25.

for it is the most prudent course of action. It now remains to the results of the power study to help identify which of the component tests are more powerful against which alternatives.

4.5 Power Study

The results of the power study for the sequential test as well as for the individual skewness and kurtosis tests serve to identify the strengths and weaknesses of the tests compared to others in the literature and provide information that assists the user in choosing the appropriate combination of significance levels for the individual tests. The first set of power studies compared the sequential test to the performance of the Anderson-Darling (A^2), Cramér-von Mises (W^2), and Kolmogorov-Smirnov (K-S) EDF tests against certain alternative distributions as documented by Bush. Next, power was calculated against a series of alternatives examined by Wozniak, although the results are not directly comparable due to the fact that her test was for the two-parameter Weibull and was based upon the extreme-value distribution. The individual skewness and kurtosis tests were then examined against both sets of alternatives to determine which tests were more powerful against certain alternatives. Finally, some experiments using one-sided versions of the separate tests and

the sequential test on a subset of alternatives were conducted to explore what kind of impact it had on power.

4.5.1 Sequential Test Power.

4.5.1.1 Power versus Bush's Alternatives. As noted in Chapter 3, Bush studied the $\beta = 1$ and $\beta = 3.5$ null hypotheses and tabled the power achieved at $\alpha = 0.05$ for each of the three EDF test statistics noted above for sample sizes $n = 5, 15$, and 25 against several different alternatives. For each null and alternate hypothesis combination Bush examined, the sequential test was evaluated for the same sample sizes with the addition of $n = 50$, and the results tabled for all combinations of significance levels for the two component tests. To compare results with Bush's findings, each of the tables that follow report the power for the sequential test for three different combinations of significance levels for the skewness and kurtosis tests that yield an attained significance level as close as possible to the $\alpha = 0.05$ used by Bush. The same set of α-level combinations are repeated for each alternative distribution to make the results truly comparable. When reading the tables, note the fact that although the three reported power results for the sequential test are aligned with the three EDF test results, this by no means restricts the comparison to those powers that are lined up horizontally. The sequential test power for each combination of individual significance levels needs to be compared to all three of the EDF powers. Also note that for both the null hypotheses, one of the tables shows the results from the verification runs where the alternate was identical to the null hypothesis. In these cases, the power should be equivalent to the attained significance level within some random error.

For the Weibull shape $\beta = 1$ hypothesis, the new test performed admirably. Against the two Beta alternatives, the sequential test actually exceeded or matched the power of the EDF tests, in several cases quite substantially. An even better performance was seen against the uniform alternative, where the sequential test scored resoundingly better than all the EDF tests for every sample size. Surprisingly, though, the test did not fare so well against the normal and gamma

Table 4.8 Sequential Test Power: H_0:Weibull($\beta = 1$); H_a:Beta(2,2).

Sample Size	EDF Test	Bush's Power	Sequential Test Power	Significance Level $\sqrt{b_1}$ test	b_2 test	Obtained
5	A^2	0.031	0.107	0.04	0.02	0.048
	W^2	0.082	0.123	0.05	0.01	0.054
	K-S	0.069	0.071	0.02	0.04	0.048
15	A^2	0.508	0.597	0.04	0.02	0.048
	W^2	0.578	0.633	0.05	0.01	0.054
	K-S	0.454	0.500	0.02	0.04	0.049
25	A^2	0.867	0.904	0.04	0.04	0.051
	W^2	0.899	0.920	0.05	0.01	0.052
	K-S	0.771	0.881	0.03	0.04	0.047
50	n/a	n/a	1.000	0.04	0.04	0.051
	n/a	n/a	1.000	0.05	0.01	0.052

Table 4.9 Sequential Test Power: H_0:Weibull($\beta = 1$); H_a:Beta(2,3).

Sample Size	EDF Test	Bush's Power	Sequential Test Power	Significance Level $\sqrt{b_1}$ test	b_2 test	Obtained
5	A^2	0.014	0.082	0.04	0.02	0.048
	W^2	0.066	0.094	0.05	0.01	0.054
	K-S	0.057	0.055	0.02	0.04	0.048
15	A^2	0.359	0.371	0.04	0.02	0.048
	W^2	0.428	0.406	0.05	0.01	0.054
	K-S	0.340	0.291	0.02	0.04	0.049
25	A^2	0.715	0.675	0.04	0.04	0.051
	W^2	0.773	0.711	0.05	0.01	0.052
	K-S	0.628	0.629	0.03	0.04	0.047
50	n/a	n/a	0.976	0.04	0.04	0.051
	n/a	n/a	0.982	0.05	0.01	0.052

distributions relative to the EDF tests. For these cases, the results indicate that the new procedure performed better than the W^2 and K-S tests for sample size $n = 5$, but failed to keep up with them for all other sample sizes. Although it lagged the EDF tests by roughly 10-15% for both the normal and gamma alternates, the sequential test still showed acceptable power against the normal case, but really struggled with the gamma. When paired with the Weibull alternatives, the sequential test was better able to distinguish for the smallest sample size (5), and it beat the K-S at $n = 15$, but could not maintain superiority for the larger sample sizes. Sequential test power lagged the EDF power by more than 20% in some cases for the $\beta = 2$ but closed the gap considerably for the $\beta = 3.5$ alternate. The verification run against the Weibull with $\beta = 1$ demonstrated good

Table 4.10 Sequential Test Power: H_0:Weibull($\beta = 1$); H_a:Normal(0,1).

Sample Size	EDF Test	Bush's Power	Sequential Test Power	Significance Level $\sqrt{b_1}$ test	b_2 test	Attained
5	A^2	0.202	0.116	0.04	0.02	0[illegible]8
	W^2	0.099	0.135	0.05	0.01	0[illegible]4
	K-S	0.079	0.073	0.02	0.04	0[illegible]8
15	A^2	0.685	0.569	0.04	0.02	0[illegible]8
	W^2	0.699	0.602	0.05	0.01	0[illegible]4
	K-S	0.606	0.482	0.02	0.04	0[illegible]9
25	A^2	0.938	0.816	0.04	0.04	0[illegible]1
	W^2	0.950	0.839	0.05	0.01	0[illegible]2
	K-S	0.904	0.790	0.03	0.04	0[illegible]7
50	n/a	n/a	0.984	0.04	0.04	0[illegible]1
	n/a	n/a	0.986	0.05	0.01	0[illegible]2

Table 4.11 Sequential Test Power: H_0:Weibull($\beta = 1$); H_a:Uniform(0,2).

Sample Size	EDF Test	Bush's Power	Sequential Test Power	Significance Level $\sqrt{b_1}$ test	b_2 test	Attained
5	A^2	0.050	0.115	0.04	0.02	0[illegible]8
	W^2	0.084	0.127	0.05	0.01	0[illegible]4
	K-S	0.072	0.088	0.02	0.04	0[illegible]8
15	A^2	0.384	0.622	0.04	0.02	0[illegible]8
	W^2	0.446	0.644	0.05	0.01	0[illegible]4
	K-S	0.328	0.564	0.02	0.04	0[illegible]9
25	A^2	0.703	0.924	0.04	0.04	0[illegible]1
	W^2	0.746	0.926	0.05	0.01	0[illegible]2
	K-S	0.575	0.909	0.03	0.04	0[illegible]7
50	n/a	n/a	1.000	0.04	0.04	0[illegible]1
	n/a	n/a	1.000	0.05	0.01	0[illegible]2

Table 4.12 Sequential Test Power: H_0:Weibull($\beta = 1$); H_a:Gamma(2,1).

Sample Size	EDF Test	Bush's Power	Sequential Test Power	Significance Level $\sqrt{b_1}$ test	b_2 test	Attained
5	A^2	0.114	0.050	0.04	0.02	0[illegible]8
	W^2	0.043	0.057	0.05	0.01	0[illegible]4
	K-S	0.040	0.042	0.02	0.04	0[illegible]8
15	A^2	0.122	0.079	0.04	0.02	0[illegible]8
	W^2	0.116	0.090	0.05	0.01	0[illegible]4
	K-S	0.099	0.064	0.02	0.04	0[illegible]9
25	A^2	0.231	0.109	0.04	0.04	0[illegible]1
	W^2	0.245	0.120	0.05	0.01	0[illegible]2
	K-S	0.196	0.094	0.03	0.04	0[illegible]7
50	n/a	n/a	0.157	0.04	0.04	0[illegible]1
	n/a	n/a	0.175	0.05	0.01	0[illegible]2

Table 4.13 Sequential Test Power: H_0:Weibull($\beta = 1$); H_a:Weibull($\beta = 1$).

Sample Size	EDF Test	Bush's Power	Sequential Test Power	Significance Level $\sqrt{b_1}$ test	b_2 test	Attained
5	A^2	0.049	0.051	0.04	0.02	[illegible]
	W^2	0.045	0.056	0.05	0.01	[illegible]
	K-S	0.045	0.051	0.02	0.04	[illegible]
15	A^2	0.043	0.047	0.04	0.02	[illegible]
	W^2	0.053	0.052	0.05	0.01	[illegible]
	K-S	0.049	0.047	0.02	0.04	[illegible]
25	A^2	0.047	0.050	0.04	0.04	[illegible]
	W^2	0.056	0.051	0.05	0.01	[illegible]
	K-S	0.057	0.046	0.03	0.04	[illegible]
50	n/a	n/a	0.049	0.04	0.04	[illegible]
	n/a	n/a	0.050	0.05	0.01	[illegible]

Table 4.14 Sequential Test Power: H_0:Weibull($\beta = 1$); H_a:Weibull($\beta = 2$).

Sample Size	EDF Test	Bush's Power	Sequential Test Power	Significance Level $\sqrt{b_1}$ test	b_2 test	Attained
5	A^2	0.049	0.069	0.04	0.02	[illegible]
	W^2	0.055	0.079	0.05	0.01	[illegible]
	K-S	0.051	0.048	0.02	0.04	[illegible]
15	A^2	0.321	0.243	0.04	0.02	[illegible]
	W^2	0.346	0.270	0.05	0.01	[illegible]
	K-S	0.277	0.184	0.02	0.04	[illegible]
25	A^2	0.655	0.411	0.04	0.04	[illegible]
	W^2	0.700	0.447	0.05	0.01	[illegible]
	K-S	0.575	0.369	0.03	0.04	[illegible]
50	n/a	n/a	0.714	0.04	0.04	[illegible]
	n/a	n/a	0.745	0.05	0.01	[illegible]

Table 4.15 Sequential Test Power: H_0:Weibull($\beta = 1$); H_a:Weibull($\beta = 3.5$).

Sample Size	EDF Test	Bush's Power	Sequential Test Power	Significance Level $\sqrt{b_1}$ test	b_2 test	Attained
5	A^2	0.052	0.113	0.04	0.02	[illegible]
	W^2	0.097	0.133	0.05	0.01	[illegible]
	K-S	0.079	0.073	0.02	0.04	[illegible]
15	A^2	0.624	0.570	0.04	0.02	[illegible]
	W^2	0.667	0.605	0.05	0.01	[illegible]
	K-S	0.568	0.481	0.02	0.04	[illegible]
25	A^2	0.930	0.842	0.04	0.04	[illegible]
	W^2	0.947	0.864	0.05	0.01	[illegible]
	K-S	0.885	0.814	0.03	0.04	[illegible]
50	n/a	n/a	0.992	0.04	0.04	[illegible]
	n/a	n/a	0.994	0.05	0.01	[illegible]

agreement between the power and the attained significance levels as hoped, indicating valid code and accurate critical values. Typical of goodness-of-fit tests, the power in all cases increased with sample size. Also note, though, that in almost all cases for this null hypothesis, the best power was achieved when the skewness test is conducted at higher significance levels. This demonstrates the importance of the proper choice of significance levels for the two tests. It also affirms several of the earlier findings that indicated that for smaller values of shape, the skewness test might be superior in discriminatory power. Recall from the scatter plots that the correlation between $\sqrt{b_1}$ and b_2 suggested one of the two would emerge as dominant, and that the smaller variability of the skewness statistic hinted it would take that dominant position. When the individual tests are examined, this relationship will become clearer.

One means to quickly summarize the results for the shape $\beta = 1$ hypothesis against Bush's alternates is in the form of a "light" chart as provided in Table 4.16. This table portrays how the sequential test performed relative to the EDF tests in a readily digestible and visual format. A clear box (□) indicates cases where the sequential test outperformed the given EDF test in at least two of the three different combinations of significance levels as given in the previous tables. A crossed box (⊞) represents those cases where the sequential test power fell below the EDF test power but by no more than 5%. A crossed box also covers those instances where the sequential power only exceeded EDF power in one of the three tabled samples. A solid box (■) shows that the sequential test power fell below EDF power by a margin greater than 5% in all three cases. Obviously, the more clear or crossed boxes, the better the sequential test performed. This format facilitates quick identification of the strengths and weaknesses of the new test by alternate distribution and sample size as well as permitting a quick comparison with each EDF test evaluated.

Table 4.16 highlights the trends discussed above and provides additional insights as well. Notice the superb performance of the sequential test for the very small sample sizes ($n = 5$) relative to the EDF tests. Note also the weaknesses in the larger sample size arena ($n = 25$).

Recall the trend observed in the scatter plots that variability in $\sqrt{b_1}$ and b_2 increased as sample size increased for the smaller values of Weibull shape. Increased variability in a test statistic leads to reduced power, which one observes here. It is important to note that a ■ does not necessarily indicate poor power; rather, it simply denotes cases where the sequential power lagged that of the EDF power. In case of the normal alternative, for example, the sequential test clearly had good power (see Table 4.10), but it just trailed the EDF power somewhat. In general, these findings are quite positive – the fact that a two-sided sequential test can actually exhibit superior power at all against such popular tests is noteworthy.

Table 4.16 Summary of Power Results for $\beta = 1$

Sample Size	5			15			25		
EDF Test	A^2	W^2	K-S	A^2	W^2	K-S	A^2	W^2	K-S
Beta(2,2)	□	□	□	□	□	□	□	□	□
Beta(2,3)	□	□	□	□	⊞	□	⊞	■	□
Normal(0,1)	■	□	□	■	■	⊞	■	■	■
Uniform(0,2)	□	□	□	□	□	□	□	□	□
Gamma(2,1)	■	□	□	■	⊞	⊞	■	■	■
Weibull(2)	□	□	□	■	■	⊞	■	■	■
Weibull(3.5)	□	□	□	⊞	■	□	■	■	⊞

□ = Sequential test power higher than EDF test power in at *least* 2 of 3 combinations

⊞ = Sequential test power within 5% of EDF test power in 2 of 3 combinations

■ = Sequential test power more than 5% lower than EDF test power in all 3 combinations

For the $\beta = 3.5$ null hypothesis, the challenge was somewhat tougher because most of Bush's alternate distributions were mound-shaped, and this Weibull distribution is a close approximation to the normal. Consequently, one finds generally lower power across the board for both the sequential test and the EDF tests, but again the sequential procedure exhibits some definite superiority to the other conventional tests. Against the two beta alternatives, the sequential test tied or surpassed the EDF tests quite consistently, but not as often against the Beta(2,3) as the Beta(2,2). As would be expected against the normal alternate, both tests had low power, but the sequential test exhibited slightly better performance. Similarly, better power was evident against the uniform alternative for

Table 4.17 Sequential Test Power: H_0:Weibull($\beta = 3.5$); H_a:Bet[illegible]2).

Sample Size	EDF Test	Bush's Power	Sequential Test Power	Significance Level		
				$\sqrt{b_1}$ test	b_2 test	Atta[illegible]d
5	A^2	0.052	0.047	0.04	0.02	0.0[illegible]
	W^2	0.054	0.052	0.03	0.04	0.0[illegible]
	K-S	0.050	0.054	0.02	0.05	0.0[illegible]
15	A^2	0.050	0.046	0.04	0.02	0.0[illegible]
	W^2	0.048	0.054	0.03	0.03	0.0[illegible]
	K-S	0.044	0.062	0.02	0.04	0.0[illegible]
25	A^2	0.073	0.039	0.04	0.01	0.0[illegible]
	W^2	0.061	0.077	0.03	0.03	0.0[illegible]
	K-S	0.067	0.092	0.02	0.04	0.0[illegible]
50	n/a	n/a	0.184	0.03	0.03	0.0[illegible]
	n/a	n/a	0.218	0.02	0.04	0.0[illegible]

Table 4.18 Sequential Test Power: H_0:Weibull($\beta = 3.5$); H_a:Bet[illegible]3).

Sample Size	EDF Test	Bush's Power	Sequential Test Power	Significance Level		
				$\sqrt{b_1}$ test	b_2 test	Atta[illegible]d
5	A^2	0.053	0.052	0.04	0.02	0.0[illegible]
	W^2	0.056	0.055	0.03	0.04	0.0[illegible]
	K-S	0.053	0.056	0.02	0.05	0.0[illegible]
15	A^2	0.065	0.056	0.04	0.02	0.0[illegible]
	W^2	0.069	0.059	0.03	0.03	0.0[illegible]
	K-S	0.061	0.062	0.02	0.04	0.0[illegible]
25	A^2	0.103	0.060	0.04	0.01	0.0[illegible]
	W^2	0.094	0.077	0.03	0.03	0.0[illegible]
	K-S	0.093	0.081	0.02	0.04	0.0[illegible]
50	n/a	n/a	0.132	0.03	0.03	0.0[illegible]
	n/a	n/a	0.134	0.02	0.04	0.0[illegible]

Table 4.19 Sequential Test Power: H_0:Weibull($\beta = 3.5$); H_a:Norm[illegible],1).

Sample Size	EDF Test	Bush's Power	Sequential Test Power	Significance Level		
				$\sqrt{b_1}$ test	b_2 test	Atta[illegible]d
5	A^2	0.054	0.050	0.04	0.02	0.0[illegible]
	W^2	0.057	0.051	0.03	0.04	0.0[illegible]
	K-S	0.053	0.050	0.02	0.05	0.0[illegible]
15	A^2	0.061	0.072	0.04	0.02	0.0[illegible]
	W^2	0.066	0.067	0.03	0.03	0.0[illegible]
	K-S	0.063	0.065	0.02	0.04	0.0[illegible]
25	A^2	0.073	0.077	0.04	0.01	0.0[illegible]
	W^2	0.071	0.082	0.03	0.03	0.0[illegible]
	K-S	0.076	0.077	0.02	0.04	0.0[illegible]
50	n/a	n/a	0.106	0.03	0.03	0.0[illegible]
	n/a	n/a	0.107	0.02	0.04	0.0[illegible]

Table 4.20 Sequential Test Power: H_0:Weibull($\beta = 3.5$); H_a:Unifor[illegible],2).

Sample	EDF	Bush's	Sequential	Significance Level		
Size	Test	Power	Test Power	$\sqrt{b_1}$ test	b_2 test	Atta[illegible]
5	A^2	0.069	0.066	0.04	0.02	0.04[illegible]
	W^2	0.069	0.075	0.03	0.04	0.05[illegible]
	K-S	0.063	0.078	0.02	0.05	0.04[illegible]
15	A^2	0.131	0.117	0.04	0.02	0.05[illegible]
	W^2	0.117	0.144	0.03	0.03	0.05[illegible]
	K-S	0.086	0.167	0.02	0.04	0.05[illegible]
25	A^2	0.232	0.179	0.04	0.01	0.04[illegible]
	W^2	0.172	0.303	0.03	0.03	0.05[illegible]
	K-S	0.129	0.345	0.02	0.04	0.05[illegible]
50	n/a	n/a	0.725	0.03	0.03	0.05[illegible]
	n/a	n/a	0.764	0.02	0.04	0.05[illegible]

Table 4.21 Sequential Test Power: H_0:Weibull($\beta = 3.5$); H_a:Gamm[illegible],1).

Sample	EDF	Bush's	Sequential	Significance Level		
Size	Test	Power	Test Power	$\sqrt{b_1}$ test	b_2 test	Atta[illegible]
5	A^2	0.207	0.100	0.04	0.02	0.04[illegible]
	W^2	0.195	0.094	0.03	0.04	0.05[illegible]
	K-S	0.180	0.090	0.02	0.05	0.04[illegible]
15	A^2	0.476	0.371	0.04	0.02	0.05[illegible]
	W^2	0.465	0.337	0.03	0.03	0.05[illegible]
	K-S	0.423	0.298	0.02	0.04	0.05[illegible]
25	A^2	0.679	0.613	0.04	0.01	0.04[illegible]
	W^2	0.652	0.578	0.03	0.03	0.05[illegible]
	K-S	0.602	0.526	0.02	0.04	0.05[illegible]
50	n/a	n/a	0.899	0.03	0.03	0.05[illegible]
	n/a	n/a	0.869	0.02	0.04	0.05[illegible]

Table 4.22 Sequential Test Power: H_0:Weibull($\beta = 3.5$); H_a:Weibull([illegible]= 1).

Sample	EDF	Bush's	Sequential	Significance Level		
Size	Test	Power	Test Power	$\sqrt{b_1}$ test	b_2 test	Atta[illegible]
5	A^2	0.241	0.162	0.04	0.02	0.04[illegible]
	W^2	0.238	0.153	0.03	0.04	0.05[illegible]
	K-S	0.219	0.143	0.02	0.05	0.04[illegible]
15	A^2	0.667	0.572	0.04	0.02	0.05[illegible]
	W^2	0.651	0.532	0.03	0.03	0.05[illegible]
	K-S	0.558	0.479	0.02	0.04	0.05[illegible]
25	A^2	0.907	0.837	0.04	0.01	0.04[illegible]
	W^2	0.881	0.810	0.03	0.03	0.05[illegible]
	K-S	0.801	0.767	0.02	0.04	0.05[illegible]
50	n/a	n/a	0.990	0.03	0.03	0.05[illegible]
	n/a	n/a	0.984	0.02	0.04	0.05[illegible]

Table 4.23 Sequential Test Power: H_0:Weibull($\beta = 3.5$); H_a:Weibull($\beta = 2$).

Sample Size	EDF Test	Bush's Power	Sequential Test Power	Significance Level $\sqrt{b_1}$ test	b_2 test	Attained
5	A^2	0.060	0.063	0.04	0.02	0.049
	W^2	0.063	0.062	0.03	0.04	0.050
	K-S	0.064	0.061	0.02	0.05	0.049
15	A^2	0.117	0.130	0.04	0.02	0.052
	W^2	0.117	0.116	0.03	0.03	0.050
	K-S	0.102	0.105	0.02	0.04	0.050
25	A^2	0.191	0.210	0.04	0.01	0.048
	W^2	0.176	0.196	0.03	0.03	0.053
	K-S	0.172	0.169	0.02	0.04	0.052
50	n/a	n/a	0.386	0.03	0.03	0.055
	n/a	n/a	0.335	0.02	0.04	0.054

Table 4.24 Sequential Test Power: H_0:Weibull($\beta = 3.5$); H_a:Weibull($\beta = 3.5$).

Sample Size	EDF Test	Bush's Power	Sequential Test Power	Significance Level $\sqrt{b_1}$ test	b_2 test	Attained
5	A^2	0.052	0.051	0.04	0.02	0.049
	W^2	0.053	0.051	0.03	0.04	0.050
	K-S	0.052	0.051	0.02	0.05	0.049
15	A^2	0.056	0.051	0.04	0.02	0.052
	W^2	0.058	0.050	0.03	0.03	0.050
	K-S	0.057	0.049	0.02	0.04	0.050
25	A^2	0.046	0.047	0.04	0.01	0.048
	W^2	0.045	0.053	0.03	0.03	0.053
	K-S	0.051	0.052	0.02	0.04	0.052
50	n/a	n/a	0.052	0.03	0.03	0.055
	n/a	n/a	0.054	0.02	0.04	0.054

all sample sizes compared with the EDF results. Unfortunately, the trend was not continued with the gamma distribution. Here the EDF tests surpassed the sequential test, but the lead was not as significant for the $n = 25$ case as it was for the smaller sample sizes. Finally, for the Weibull cases, the sequential test outpaced the EDF tests on the $\beta = 2$ alternate, but did not fare as well against the $\beta = 1$ except for larger sample sizes. Note, however, that in all cases where the sequential test failed to exceed the EDF test power, it was almost always within 10%. This is significant, especially considering the fact that this new test requires substantially less computation to conduct.

Again, for all alternates the power increased with sample size, and for the $\beta = 3.5$ null hypothesis, the sequential test performed consistently better across all sample sizes, whereas with

Table 4.25 Summary of Power Results for $\beta = 3.5$

Sample Size	5			15			25		
EDF Test	A^2	W^2	K-S	A^2	W^2	K-S	A^2	W^2	K-S
Beta(2,2)	□	□	□	□	□	□	□	□	□
Beta(2,3)	□	□	□	⊞	⊞	⊞	⊞	⊞	⊞
Normal(0,1)	□	⊞	□	□	□	□	□	□	□
Uniform(0,2)	□	□	□	□	□	□	□	□	□
Gamma(2,1)	■	■	■	■	■	■	■	■	⊞
Weibull(2)	□	□	⊞	□	□	□	□	□	□
Weibull(1)	■	■	■	■	■	⊞	■	■	□

□ = Sequential test power higher than EDF test power in at *least* 2 of 3 combinations

⊞ = Sequential test power within 5% of EDF test power in 2 of 3 combinations

■ = Sequential test power more than 5% lower than EDF test power in all 3 combinations

the $\beta = 1$ null hypothesis, it seemed to lose its lead as sample size increased. Additionally, one finds here that the sequential test performed best in several situations where the kurtosis test was conducted at higher significance levels (except for the gamma and Weibull alternates). This indicates that the kurtosis test dominates the skewness test against particular alternatives with the $\beta = 3.5$ hypothesis, a reverse of the finding for $\beta = 1$. These facts lend further import to the proper selection of α-levels for the two tests. Additionally, it is clear that both of the component tests have their strengths and weaknesses, which enhances the appeal of this sequential procedure.

A light chart for the $\beta = 3.5$ case is given in Table 4.25, which summarizes the above findings. It is clear that the sequential test performed better for this larger value of shape relative to the EDF tests than it did for the small $\beta = 1$ shape. Again recall from the scatterplots that the test statistic variability diminished as shape and sample size increased. This translates directly to better power here. The frequency with which the sequential procedure outperformed the EDF tests is quite remarkable and certainly marks an important finding.

The results of this part of the power study against the alternate distributions used by Bush are quite encouraging. The sequential test based on skewness and kurtosis demonstrated equivalent or

better power than the popular and much more computationally expensive EDF tests in a majority of cases. The key weaknesses compared to the EDF tests were against the gamma, normal, and some of the Weibull alternatives for both null hypotheses. Picking higher significance levels for the skewness test generally improved power for the $\beta = 1$ hypothesis, whereas higher levels for the kurtosis test seemed to increase power for the $\beta = 3.5$ case against the symmetric alternates. The particular strengths of each component test will emerge in the following power study. This sequential test, then, has demonstrated some clear value as a new tool in the goodness-of-fit field.

4.5.1.2 Power Versus Wozniak's Alternatives. Comparing the sequential test results to those obtained by Wozniak is not as straight forward as with Bush's findings, but the selection of alternates commonly used to evaluate goodness-of-fit tests for the Weibull makes the results nonetheless valuable for future comparative studies. Since Wozniak premised her study on evaluating the two-parameter Weibull using an extreme-value goodness-of-fit test that estimated parameters, her power results are not directly comparable to those in this study. The fact that the current study accounts for a location parameter is significant because to address data with a nonzero location parameter, the procedure Wozniak employed requires estimation of this parameter first. It then must be subtracted from all elements of the sample, resulting in the loss of one data point, when using the typical MLE of the location parameter – the first order statistic. The resultant reduction in sample size could be very expensive if one is working with very small sample sizes. More importantly, if the location parameter is simply assumed to be zero when it is not, extreme-value tests will tend to reject Weibull data erroneously, even when the location parameter is less than unity. An example of this can be seen in the power results from Lockhart, Reilly, and Stephen's extreme-value test based on normalized sample spacings, which exhibited very high power against Weibulls with various location parameters [55: 419]. (Although this test is dissimilar to Wozniak's, it illustrates the principle.) The location- and scale-free nature of the sequential test is advantageous for these situations.

Table 4.26 Sequential Test Power: H_0:Weibull($\beta = 0.5$); H_a:$\chi^2(1)$.

Sample Size	EDF Test	Wozniak's Power	Sequential Test Power	Significance Level $\sqrt{b_1}$ test	b_2 test	Attained
20	A^2	0.0762	0.116	0.05	0.01	0.051
	W^2	0.0647	0.117	0.05	0.03	0.052
	K-S	0.0672	0.094	0.01	0.05	0.051
30	A^2	0.0932	0.153	0.05	0.01	0.051
	W^2	0.0803	0.154	0.05	0.04	0.053
	K-S	0.0745	0.122	0.01	0.05	0.051
50	n/a	n/a	0.233	0.05	0.01	0.051
	n/a	n/a	0.234	0.05	0.04	0.052

Another key difference between Wozniak's study and this one is that the log conversion to extreme-value data eliminates the shape parameter in her studies. So, to generate results that would be roughly comparable, this study used known shapes as close as possible to those that samples from the given distributions exhibited using the Weibull MLE estimate for shape as listed in table 3.6. To make a direct comparison, Wozniak would have had to operate from a known scale hypothesis in the extreme-value test. Since she was not concerned with shape but rather estimated the extreme-value scale parameter (transformed Weibull shape), the use of the nearest shape within a half-unit interval by this current power study essentially exaggerates the discrepancy between the null hypothesis and the data, leading to possible bias in favor of the sequential test. Consequently, the use of Wozniak's power results in the tables to follow should only be taken as a general reference to a related but not equivalent test. The powers reported for the sequential test are no less valuable, however, because they do quantify the test's performance against some recognized benchmarks.

The Monte Carlo runs for the power study generated results for all combinations of significance levels of the $\sqrt{b_1}$ and b_2 tests from $\alpha = 0.01(0.01)0.20$ and sample sizes $n = 20, 30$, and 50 for each alternate distribution. The full results are tabled in the Appendix, but specific powers for the sequential test and Wozniak's EDF tests at $\alpha = 0.05$ are reported in Tables 4.26 to 4.35. (Wozniak did not document the $n = 50$ powers, however.)

The first of the alternatives examined here was the $\chi^2(1)$. Since the nearby $\chi^2(2)$ is a Weibull($\beta = 1$) distribution, and the shape estimate found in Table 3.6 was roughly 0.6, the

Table 4.27 Sequential Test Power: H_0:Weibull($\beta = 1$); H_a:$\chi^2(1)$.

Sample Size	EDF Test	Wozniak's Power	Sequential Test Power	Significance Level $\sqrt{b_1}$ test	b_2 test	Attained
20	A^2	0.0762	0.089	0.05	0.01	0.054
	W^2	0.0647	0.079	0.04	0.03	0.050
	K-S	0.0672	0.087	0.01	0.05	0.054
30	A^2	0.0932	0.098	0.05	0.01	0.050
	W^2	0.0803	0.085	0.04	0.04	0.050
	K-S	0.0745	0.088	0.01	0.05	0.051
50	n/a	n/a	0.111	0.05	0.01	0.052
	n/a	n/a	0.096	0.04	0.04	0.051

Table 4.28 Sequential Test Power: H_0:Weibull($\beta = 1.5$); H_a:$\chi^2(4)$.

Sample Size	Sequential Test Power	Significance Level $\sqrt{b_1}$ test	b_2 test	Attained
20	0.079	0.05	0.01	0.055
	0.069	0.04	0.02	0.050
	0.067	0.02	0.04	0.050
30	0.084	0.05	0.01	0.054
	0.075	0.04	0.02	0.048
	0.071	0.02	0.04	0.049
50	0.104	0.05	0.01	0.053
	0.088	0.03	0.04	0.051

null hypotheses used were the $\beta = 0.5$ and $\beta = 1.0$. From the results in Table 4.26 and 4.27, it is interesting to note that while the power for all tests was generally low, the sequential test had better power for the $\beta = 0.5$ hypothesis than the $\beta = 1.0$, in spite of the fact that the $\chi^2(1)$ is more similar to the Weibull shape 0.5. One would expect lower power against a more similar distribution. The sequential test registered better power than the EDF extreme-value results across the board.

For the $\chi^2(4)$ alternative, the shape estimates from Table 3.6 were approximately 1.6, leading to the use of a $\beta = 1.5$ null hypothesis for the sequential test. Unfortunately, Wozniak's paper omitted her power results for this alternative; thus, there are no values to compare with. She did, however, comment that power was weaker for the $\chi^2(4)$ alternate than for the $\chi^2(1)$, with no power greater than 10% even with a sample size of 50 [77: 159]. The same could almost be said for the sequential test seen in Table 4.28; it demonstrated less power against this distribution than for the $\chi^2(1)$, but it did achieve slightly better than 10% power for the $n = 50$ case.

Table 4.29 Sequential Test Power: H_0:Weibull($\beta = 1$); H_a:Lognormal(0,1).

Sample Size	EDF Test	Wozniak's Power	Sequential Test Power	Significance Level $\sqrt{b_1}$ test	b_2 test	Attained
20	A^2	0.2156	0.156	0.05	0.01	0.054
	W^2	0.2059	0.140	0.04	0.03	0.050
	K-S	0.1661	0.149	0.01	0.05	0.054
30	A^2	0.3335	0.197	0.05	0.01	0.050
	W^2	0.3031	0.180	0.04	0.04	0.050
	K-S	0.2199	0.179	0.01	0.05	0.051
50	n/a	n/a	0.270	0.05	0.01	0.052
	n/a	n/a	0.248	0.04	0.04	0.051

Turning to the lognormal alternative, the nearest Weibull shape to use for the null hypothesis was the Weibull($\beta = 1$), given that the shape estimates from Table 3.6 were roughly 1.1. The sequential test yielded fairly low power against this distribution and was soundly outperformed by the EDF test results (see Table 4.29). Using high significance levels for the skewness test appeared to yield better power than a high level for the kurtosis test.

For each of the transformed alternate distributions utilized by Wozniak, the $\beta = 0.5$ and $\beta = 1$ Weibull null hypotheses were used. Referring again to Table 3.6, the transformed logistic samples produced shape estimates near 0.6, the double exponential near 0.8, and the Cauchy near 0.4. For the logistic case, power was considerably weaker than the EDF tests for the $\beta = 0.5$ hypothesis, but somewhat better for the $\beta = 1$, which is not surprising because the higher shape value is a greater departure from the estimate of 0.6. Power against the double exponential was very close to that of the EDF tests for $\beta = 1$, but dramatically less for the $\beta = 0.5$ case. The best power against the transformed alternatives was found against the Cauchy, but it was still insufficient to better that of the extreme-value EDF results. Again, one must keep in mind that the comparisons are not with equivalent tests; they are only useful insomuch as they give ballpark figures for the performance of the commonly used tests.

4.5.2 Individual Skewness and Kurtosis Test Results. For every combination of null and alternate hypotheses discussed in the previous section, another power study was conducted using the

Table 4.30 Sequential Test Power: H_0:Weibull($\beta = 0.5$); H_a:XLogistic(0,1).

Sample Size	EDF Test	Wozniak's Power	Sequential Test Power	Significance Level $\sqrt{b_1}$ test	b_2 test	Attained
20	A^2	0.3109	0.185	0.05	0.01	0.051
	W^2	0.3002	0.186	0.05	0.03	0.052
	K-S	0.2375	0.181	0.01	0.05	0.051
30	A^2	0.4602	0.221	0.05	0.01	0.051
	W^2	0.4392	0.221	0.05	0.04	0.053
	K-S	0.3382	0.217	0.01	0.05	0.051
50	n/a	n/a	0.272	0.05	0.01	0.051
	n/a	n/a	0.273	0.05	0.04	0.052

Table 4.31 Sequential Test Power: H_0:Weibull($\beta = 1$); H_a:XLogistic(0,1).

Sample Size	EDF Test	Wozniak's Power	Sequential Test Power	Significance Level $\sqrt{b_1}$ test	b_2 test	Attained
20	A^2	0.3109	0.498	0.05	0.01	0.054
	W^2	0.3002	0.475	0.04	0.03	0.050
	K-S	0.2375	0.463	0.01	0.05	0.054
30	A^2	0.4602	0.637	0.05	0.01	0.050
	W^2	0.4392	0.610	0.04	0.04	0.050
	K-S	0.3382	0.563	0.01	0.05	0.051
50	n/a	n/a	0.814	0.05	0.01	0.052
	n/a	n/a	0.791	0.04	0.04	0.051

skewness and kurtosis tests independently to help quantify which tests were most powerful against specific alternates and to provide insights that would lead to strategies for picking the appropriate combinations of significance levels for the two tests when used sequentially. The results of each of these studies are tabled in Appendix E. In order to effectively compare the results with those of the sequential tests, the resultant powers for that test and the separate skewness and kurtosis tests

Table 4.32 Sequential Test Power: H_0:Weibull($\beta = 0.5$); H_a:XDouble Exponential.

Sample Size	EDF Test	Wozniak's Power	Sequential Test Power	Significance Level $\sqrt{b_1}$ test	b_2 test	Attained
20	A^2	0.4694	0.201	0.05	0.01	0.051
	W^2	0.4702	0.201	0.05	0.03	0.052
	K-S	0.4002	0.186	0.01	0.05	0.051
30	A^2	0.6535	0.229	0.05	0.01	0.051
	W^2	0.6547	0.229	0.05	0.04	0.053
	K-S	0.5614	0.219	0.01	0.05	0.051
50	n/a	n/a	0.276	0.05	0.01	0.051
	n/a	n/a	0.276	0.05	0.04	0.052

Table 4.33 Sequential Test Power: H_0:Weibull($\beta = 1$); H_a:XDouble Exponential.

Sample Size	EDF Test	Wozniak's Power	Sequential Test Power	Significance Level		
				$\sqrt{b_1}$ test	b_2 test	Attained
20	A^2	0.4694	0.473	0.05	0.01	0.054
	W^2	0.4702	0.449	0.04	0.03	0.050
	K-S	0.4002	0.442	0.01	0.05	0.054
30	A^2	0.6535	0.613	0.05	0.01	0.050
	W^2	0.6547	0.585	0.04	0.04	0.050
	K-S	0.5614	0.547	0.01	0.05	0.051
50	n/a	n/a	0.792	0.05	0.01	0.052
	n/a	n/a	0.770	0.04	0.04	0.051

Table 4.34 Sequential Test Power: H_0:Weibull($\beta = 0.5$); H_a:XCauchy(0,1).

Sample Size	EDF Test	Wozniak's Power	Sequential Test Power	Significance Level		
				$\sqrt{b_1}$ test	b_2 test	Attained
20	A^2	0.8752	0.667	0.05	0.01	0.051
	W^2	0.8764	0.667	0.05	0.03	0.052
	K-S	0.846	0.663	0.01	0.05	0.051
30	A^2	0.9676	0.751	0.05	0.01	0.051
	W^2	0.9666	0.751	0.05	0.04	0.053
	K-S	0.9486	0.748	0.01	0.05	0.051
50	n/a	n/a	0.823	0.05	0.01	0.051
	n/a	n/a	0.823	0.05	0.04	0.052

Table 4.35 Sequential Test Power: H_0:Weibull($\beta = 1$); H_a:XCauchy(0,1).

Sample Size	EDF Test	Wozniak's Power	Sequential Test Power	Significance Level		
				$\sqrt{b_1}$ test	b_2 test	Attained
20	A^2	0.8752	0.830	0.05	0.01	0.054
	W^2	0.8764	0.819	0.04	0.03	0.050
	K-S	0.846	0.807	0.01	0.05	0.054
30	A^2	0.9676	0.913	0.05	0.01	0.050
	W^2	0.9666	0.906	0.04	0.04	0.050
	K-S	0.9486	0.887	0.01	0.05	0.051
50	n/a	n/a	0.929	0.05	0.01	0.052
	n/a	n/a	0.928	0.04	0.04	0.051

were plotted by significance level for each sample tested. The resulting charts, grouped according to the Weibull null hypothesis examined, shed more light on the behavior and usefulness of the tests.

For the $\beta = 1$ assumption, the results were all very similar to those depicted in Figure 4.15 showing power against the Beta(2,2) alternate. It is readily apparent that the skewness test dominates the kurtosis test in terms of power. This was true for every alternate distribution considered for this null hypothesis. The jagged lines indicating the sequential test power illustrate the fact that for a given attained significance level, the power of the sequential test fluctuates considerably depending on what combination of significance levels one chooses for the individual tests. Here, since the skewness test is more powerful, choosing a higher level for it and a correspondingly lower level for kurtosis, will yield higher power, while picking a higher for the weaker kurtosis test results in lower power at the same overall significance level.

Although the skewness test excels among this collection of alternates, notice that for those specific distributions that were very similar in shape to the Weibull($\beta = 1$), the difference between the two tests was substantially less. Specifically consider the $\chi^2(1)$ and the lognormal alternatives in Figure 4.16 which shows the powers for sample size 30. Both of these PDFs resemble the Weibull($\beta = 1$) closer than the others, and both tests here perform nearly identically well. The skewness test's advantage is minimal. One can argue, then, that for alternates that share similar skewed properties, the kurtosis test can be valuable as well.

Note also that the sequential test power is bound by the power of the individual tests for all these results. This condition is due to the fact that at a fixed attained significance level, the corresponding levels for the individual tests are normally less than or equal to the overall α-level, resulting in less power than the best test individually at that same level but better power than the worst. In other words, if α_1 and α_2 are the significance levels for the skewness and kurtosis tests respectively that yield a significance of α when used sequentially, one notes from the contour

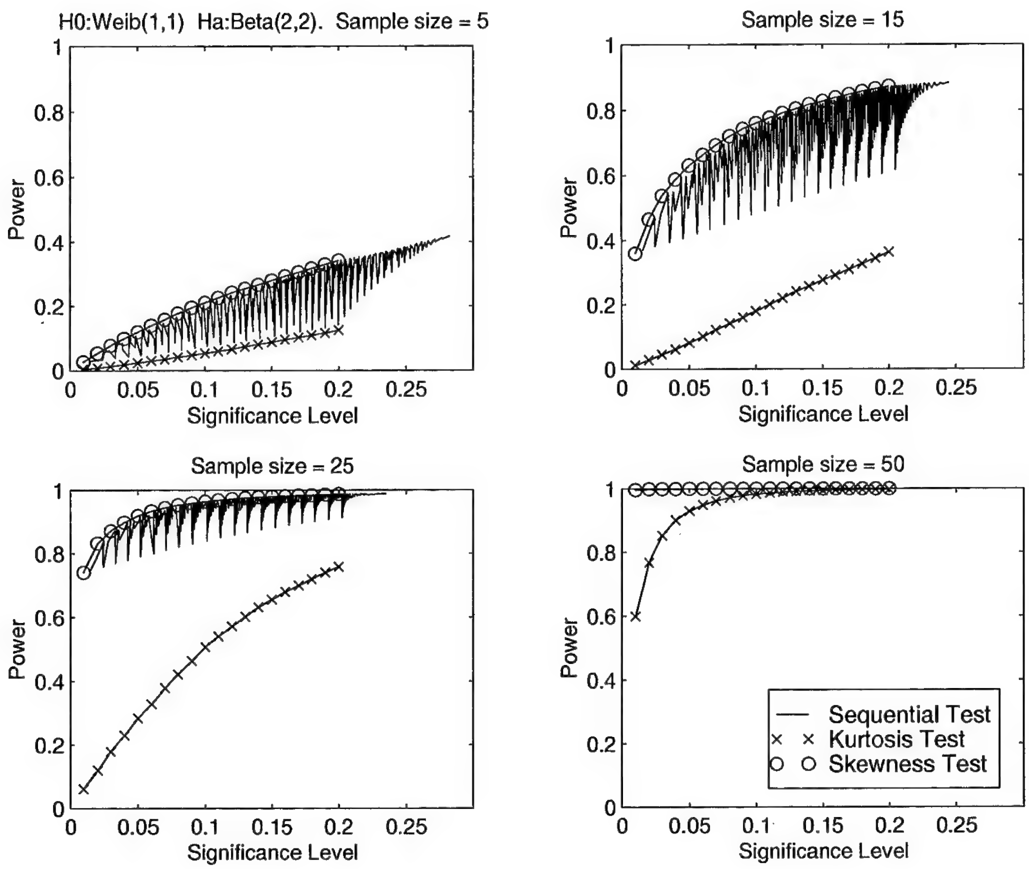


Figure 4.15 Individual vs. Sequential Power: H_0:Weibull($\beta = 1$); H_a:Beta(2,2).

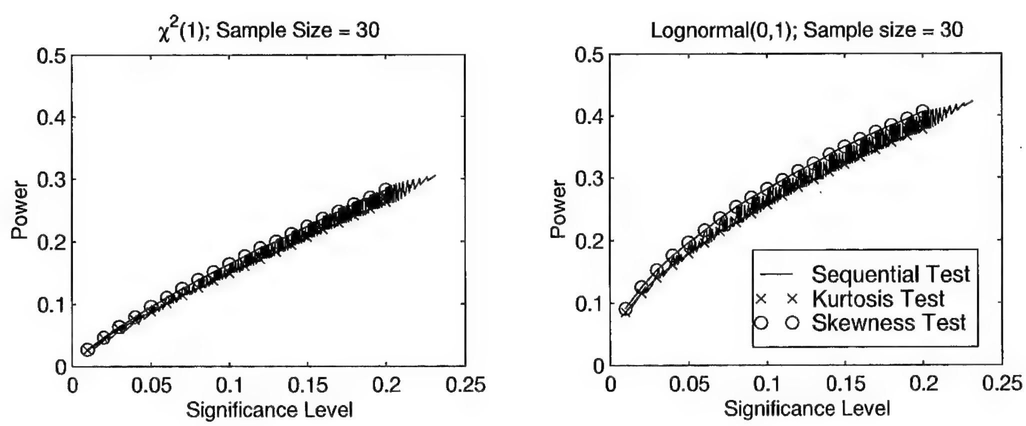


Figure 4.16 Individual vs. Sequential Power: H_0:Weibull($\beta = 1$); H_a : $\chi^2(1)$ and Lognormal(0,1); Sample Size = 30.

plots that usually both $\alpha_1 \leq \alpha$ and $\alpha_2 \leq \alpha$. Consequently, the power of both tests is less than what they would be had they been conducted at the level α. This normally keeps the sequential test power lower than its best component test at α. Further, the complementary nature of the two tests working together normally prevents the sequential test power from slipping below that of the weaker test. This bounding effect is not always present, however. There were a few cases where the sequential test power seemed to dip below that of both component tests (see the $\chi^2(1)$ and Cauchy results for $\beta = 1$); however, upon closer examination of the results, these cases were a more a result of the variability in the third decimal place of the power estimates than any real significant reduction in power. On the positive side, there are cases where the sequential test power actually exceeds that of both the tests it employs due to a particularly effective complementary relationship between the tests.

Only one such case was observed in this study, and that was with the $\beta = 3.5$ null hypothesis against the Beta(2,3) alternative for large sample sizes (25 and 50), as seen in Fig 4.17. Against this particular alternative, both the $\sqrt{b_1}$ and b_2 test have nearly equivalent power, and they obviously complement one another to the point that when employed in sequence against a sample, they reject more often than each test does independently. One would like to observe this situation more frequently, but the lag in significance levels (and the corresponding lag in power) between the component tests when used sequentially and when employed individually is apparently only overcome when both tests have nearly identical power.

Examining the remaining results for the $\beta = 3.5$ null hypothesis, one finds more mixed results than those for the $\beta = 1$ case. Here, the kurtosis demonstrates some superiority over the skewness test against several of the Bush alternatives, namely, both betas and the uniform distribution shown in Figure 4.18. Both tests performed nearly identically well against the normal distribution. Evidently, for this mound-shaped Weibull curve, kurtosis becomes more a discriminator against similarly-shaped alternatives. This is not surprising because the Weibull($\beta = 3.5$), Beta, Uniform,

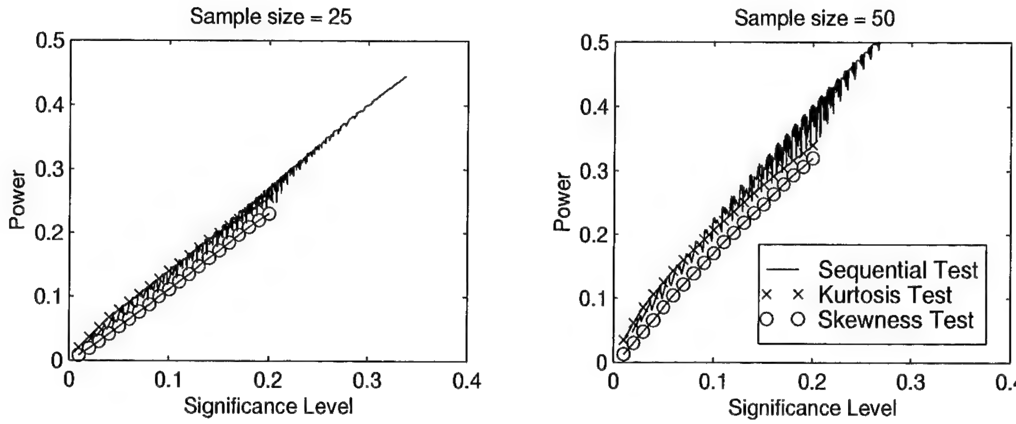


Figure 4.17 Individual vs. Sequential Power: H_0:Weibull($\beta = 3.5$); H_a:Beta(2,3).

and Normal distributions are all more or less symmetric, making their skewness very similar. It also agrees with the earlier finding that the reduced correlation between the two test statistics for larger shapes may lead to less dominance of one over the other. In contrast, the skewness test outperformed the kurtosis test against the more skewed alternatives Wozniak examined (see the plots in the Appendix D). These observations illustrate the potential value in this sequential procedure. The skewness and kurtosis tests each have their own particular strengths, and when confronted with a broad field of alternatives, they are enhanced by working together. The skewness test is useful in distinguishing among skewed alternatives generally, and the kurtosis test comes into its own when the alternatives are nearly symmetric or very similarly skewed. These are not particularly stunning observations, but the fact that empirical results confirmed the initial expectations is encouraging.

4.5.3 Directional Skewness and Kurtosis Test Power. The power study results from the individual $\sqrt{b_1}$ and b_2 tests help identify which of the two tests are most powerful for particular combinations of null and alternate hypotheses. These tests in their original form were two-sided statistical tests. Using one-sided versions of these same tests would certainly improve their power; so, to quantify these gains, several additional power studies were undertaken. Of course, using the one-sided tests makes them directional, implying one has some specific knowledge about the third

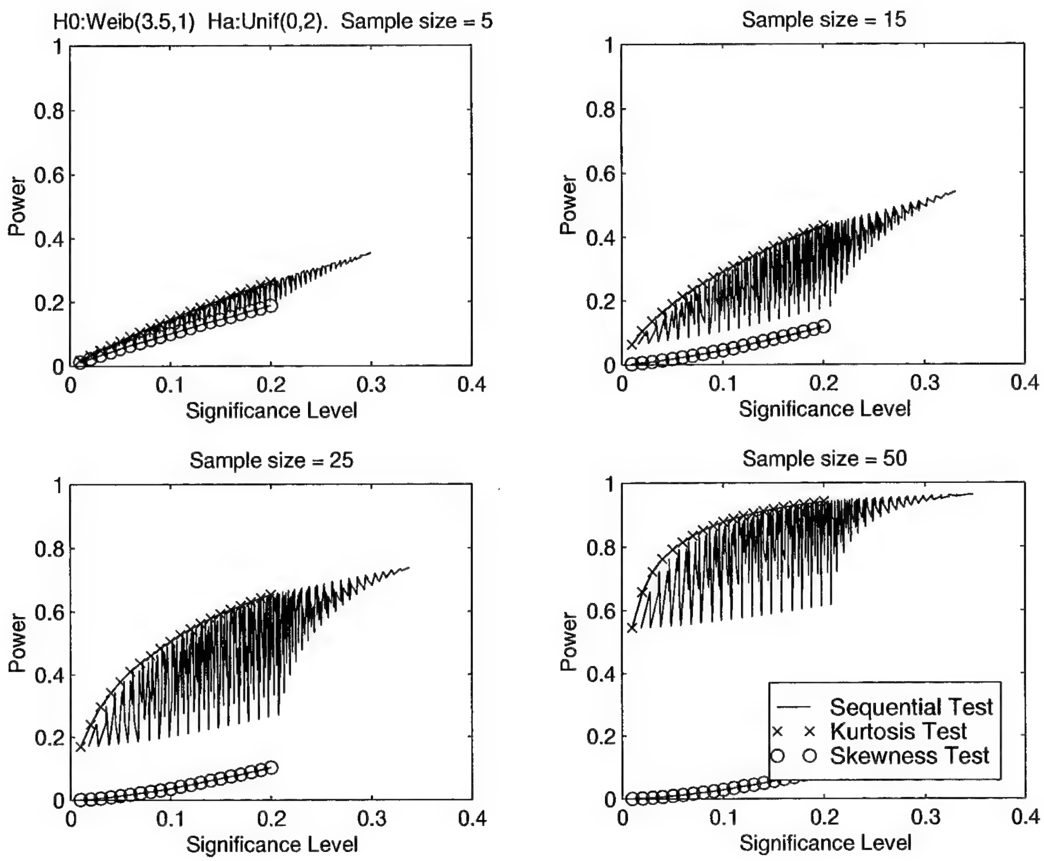


Figure 4.18 Individual vs. Sequential Power: H_0:Weibull($\beta = 3.5$); H_a:Uniform(0,2).

Table 4.36 One- and Two-Sided Power for b_2 Test: H_0:Weibull(1); H_a: Weibull(3.5).

Significance Level	Test Version	Sample Size 5	15	25	50
0.01	Two-sided	0.003	0.004	0.012	0.129
	One-sided	0.005	0.010	0.028	0.237
0.10	Two-sided	0.044	0.080	0.208	0.686
	One-sided	0.084	0.195	0.435	0.869
0.20	Two-sided	0.108	0.194	0.429	0.868
	One-sided	0.195	0.437	0.722	0.970

or fourth moments of the alternate hypotheses. This information will not always be available, but determining how beneficial such knowledge might be was worthy of investigation.

This study examined one-sided test results only for the $\beta = 1$ and $\beta = 3.5$ null hypotheses and only a subset of the alternates as noted in Table 3.11. The tables reporting the power results are presented in Appendix F, with some relevant findings discussed here. As expected, the one-sided tests produced improved power, in some cases, dramatically so. The gains were most noticeable against distributions where the particular test logically should have performed well, but did not do so in its original two-sided form. Consider, for instance, the kurtosis test's performance against the Weibull($\beta = 3.5$) alternative for the Weibull($\beta = 1$) null hypothesis as shown in Table 4.36 above. The true kurtosis of the two distributions is substantially different (9 versus 2.7127), and intuitively one would expect the original test to demonstrate good power. It did have acceptable power, but the lower tail one-sided version improved this power more than 90%. Most improvements were not as exemplary, but were nonetheless significant.

An example of the skewness test performance for the same null and alternate distributions are given in Table 4.37. The improvements for the kurtosis test varied from 10% to 96% depending on the combination of distributions being compared. The skewness test showed less dramatic improvements, ranging from 12% to 79%. Since not all null and alternate hypotheses were evaluated, conclusive remarks are not possible; nevertheless, there were notable gains in power for the separate tests when the appropriate one-sided variants were used.

Table 4.37 One- and Two-Sided Power for $\sqrt{b_1}$ Test: H_0:Weibull(1); H_a: Weibull(3.5).

Significance Level	Test Version	Sample Size 5	15	25	50
0.01	Two-sided	0.029	0.378	0.694	0.974
	One-sided	0.058	0.464	0.774	0.983
0.10	Two-sided	0.224	0.717	0.919	0.997
	One-sided	0.332	0.827	0.960	0.999
0.20	Two-sided	0.348	0.827	0.959	0.999
	One-sided	0.514	0.915	0.985	1.000

4.5.4 Directional Variants of the Sequential Test. The next logical step was to examine the impact of incorporating one-sided skewness and kurtosis tests into the sequential procedure and measure the improvements in power. Again, such modifications would require a substantial degree of prior information on the alternates one is considering, and certainly, such occasions would be rare. Consequently, only a preliminary inquiry was conducted in this direction. Also complicating this approach is the fact that each modification to the sequential test requires a reevaluation of the attained significance levels. Given these facts, a power study on directional variants of the sequential test was only accomplished for the Weibull(β = 1) null hypothesis against the fairly challenging lognormal alternative. Three different modifications were assessed: one using an upper-tail skewness test, another with only the kurtosis test in a one-tailed format, and a final one with both tests conducted in the upper-tail manner. As noted, for each of these runs, new attained significance levels had to first be determined. Contour plots or tables of these results are not reproduced in this report. Appendix G contains a set of power plots to compare these variants with the original test. Table 4.38 compares the results of these modified sequential tests to those of the original version for a specific case of $\alpha = 0.05$ with nearly identical combinations of significance levels for the individual tests. Recall that in their original form, both the skewness and kurtosis tests had similar power against the lognormal distribution, with the $\sqrt{b_1}$ test being slightly superior. The data in the table indicates the one-sided skewness test contributes substantially more power than the kurtosis test, but when both one-sided versions are used, the resulting power is significantly better than the original two-sided version. Hence, if enough information is available to

Table 4.38 Power for Sequential Test Variants: H_0:Weibull(β= 1); H_a: Lognormal(0,1).

		Sequential Test Variant			
	$\sqrt{b_1}$ Test	2-sided	Upper Tail	2-sided	Upper Tail
	b_2 Test	2-sided	2-sided	Upper Tail	Upper Tail
Size	20	0.140	0.188	0.158	0.209
Size	30	0.180	0.247	0.197	0.271
Size	50	0.248	0.332	0.265	0.365

the analyst to justify using directional versions of the $\sqrt{b_1}$ and/or b_2 tests in the sequential format, doing so can dramatically improve power. It is assumed however, that such instances would be rare; consequently, further investigation was considered beyond the scope of this effort.

4.6 Conclusion

The numerous Monte Carlo simulations, then, have generated the means to effectively employ and evaluate this new moment-based sequential goodness-of-fit test for the Weibull distribution. By first examining the distribution of the $\sqrt{b_1}$ and b_2 values from the Monte Carlo samples, some initial observations on their correlation and variability were made. Subsequently, the critical values for the two test statistics, with a demonstrated two decimal place accuracy, were tabled, providing the means to conduct the individual skewness and kurtosis tests. Closer examination of these values confirmed some of the early indications seen in the $(\sqrt{b_1}, b_2)$ scatter plots, particularly those relating to changes in variability with increasing shape and sample size. The critical values alone, though, were not sufficient to utilize the test. The attained significance levels were also necessary. These were found as well and presented in the form of contour plots that simplify the table look-up approach used in other sequential procedures. These plots yielded useful insights regarding the importance of choosing appropriate significance levels for the individual tests to optimize overall power. Of course, to make these decisions, some idea of the power of the two tests was required. A broad collection of power studies provided this information. They also indicated that the sequential test performed suprisingly well when compared to several popular EDF tests, in many cases surpassing their power at a substantially lower computational expense.

The separate $\sqrt{b_1}$ and b_2 tests exhibited various complementary strengths and weaknesses, with the skewness test demonstrating more power than that for kurtosis against the majority of alternatives considered. The kurtosis test, though, emerged superior in distinguishing among symmetric or similarly skewed distributions. These findings facilitate the proper selection of significance levels for the two tests when used sequentially. Finally, one-sided versions of the individual tests and the sequential procedure were examined to address those cases when sufficient information exists on the alternates to warrant the use of the upper or lower tailed versions of the test(s). These sequential variants demonstrated substantial enhancements in power.

V. CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

The field of goodness-of-fit tests for the Weibull distribution is a rich one; unfortunately, most tests require a moderate to high degree of computational complexity and do not deal with the three-parameter form of the distribution. This research presents an entirely new test that addresses both of these limitations. The moment-based sequential procedure discussed in this paper provides a computationally simple means of testing for three-parameter known shape Weibull distributions and demonstrates notably good power when compared to the popular and more complex EDF tests commonly used today.

The sequential procedure has several advantages in that it is invariant to location and scale parameters, requires no transformations of the data, and eliminates the need for parameter estimation. In the field of reliability theory, accounting for the location parameter permits much greater flexibility in testing data that may indeed have a minimum guaranteed life greater than zero or possibly fail on the shelf. The majority of Weibull goodness-of-fit tests are based upon the extreme-value distribution and assume a location parameter of zero. To address a non-zero location parameter, they must typically sacrifice data to estimate it. They must also log-transform the raw data before testing it. The use of tests for sample skewness and kurtosis applied in sequence, however, eliminates these steps. Moreover, because the test statistics are mere sample moments, no problematic parameter estimation techniques are necessary. In fact, most basic statistical software packages provide these values already in their summary statistics routines. Such moment-based tests are not new; there is an extensive body of literature on such tests for the normal distribution. None, however, are documented for the Weibull distribution. Additionally, the use of sequential tests is still relatively rare, but they offer the potential of more robust performance against a wide range of alternatives distributions by combining test statistics with complementary power characteristics.

Given these advantages, a sequential test for the Weibull based on sample skewness and kurtosis was developed, implemented and tested against data from 13 alternate distributions. Specifically, the test was built for values of shape $\beta = 0.5(0.5)4$ and sample sizes $n = 5(5)50$. The critical values for both the two-sided $\sqrt{b_1}$ and b_2 tests that constitute the sequential procedure were found through Monte Carlo simulation for $\alpha = 0.005(0.005)0.10, 0.10(0.01)0.20, 0.80(0.01)0.90$, and $0.90(0.005)0.995$. Their validity was affirmed both through assessments of their variability and with power studies using a true null hypothesis resulting in power equivalent to the significance level of the test.

Because the test is sequential, the attained significance levels for all the potential combinations of the two tests were likewise derived through Monte Carlo simulation. The generation of contour plots to depict these attained significance levels provided a simpler and more user-friendly means of determining the appropriate levels for the component tests, and represents a substantial improvement over earlier tabular formats. In addition, the plots yield tactical insights into the proper selection of significance levels for the two tests such that potentially higher power can be achieved.

In terms of power, the ability of the sequential test to discriminate among various distributions is very contingent upon the selection of significance levels for the individual tests. Because both the skewness and kurtosis tests have particular strengths and weaknesses, choosing the higher significance level for the more powerful of the two tests results in optimum power for the sequential procedure. But, with a mix of levels for both tests, it nevertheless demonstrates good to excellent power when compared to the popular Kolmogorov-Smirnov, Cramér-von Mises and Anderson-Darling EDF test statistics. Power studies conducted here examined the $\beta = 1$ and $\beta = 3.5$ Weibull null hypotheses against the alternatives addressed by Bush's work and the $\beta = 0.5, 1$ and 1.5 nulls against several common benchmark alternatives documented by Wozniak.

With an hypothesized shape of $\beta = 1$, the sequential test surpassed the EDF statistics for small samples ($n = 5$) against all alternatives. Against the beta and uniform distributions, it substantially outperformed the EDF tests for larger samples, but did not fare as well against the normal and gamma alternates. Among the Weibull alternatives, the sequential test performed fairly well against the $\beta = 3.5$ case, but lagged behind the EDF tests for the $\beta = 2$ case.

For the near-normal $\beta = 3.5$ hypothesis, the sequential test likewise performed well in competition with the EDF tests. It matched or beat the EDF powers for the betas, normal, and uniform alternatives, but again lagged on the gamma case. With the $\beta = 2$ Weibull alternate, the sequential test slightly exceeded the EDF tests power, but interestingly had less power against the $\beta = 1$ Weibull samples. With very small sample sizes ($n = 5$), the sequential test matched the power of the EDF tests except for the gamma and Weibull $\beta = 1$ cases. Overall, the power results for this much less involved test were remarkably good; matching and even bettering the EDF tests was quite an achievement.

While the outcomes against Wozniak's alternates are not directly comparable, the trends matched her EDF results fairly closely. Like them, the sequential test showed very low power against the two chi-squared alternates and low power versus the lognormal distribution. Moderate ability was evidenced with the transformed logistic and double exponential samples. The best power was achieved against the Cauchy data, just as in Wozniak's study [77: 160]. The moment-based test, then, was a consistently good performer when evaluated with reference to the popular EDF test statistics.

Further power studies investigated the discriminatory ability of the individual skewness and kurtosis tests to help identify their strengths and weaknesses. In general, the skewness test dominated the kurtosis test, particularly for the $\beta = 1$ Weibull hypothesis. However, for distinguishing among symmetric or lower-magnitude skewness alternatives, the b_2 test emerged superior. The kurtosis test closely matched the power of the skewness test in cases where there was only a fine

distinction between the null and alternate hypotheses, such as between the Weibull $\beta = 3.5$ and the normal, or between the Weibull $\beta = 1$ and the lognormal distribution. Thus, the complementary nature of the two tests was clear, reinforcing the utility of a sequential procedure. For situations where some information on the alternatives to consider does exist, identifying which test should be more powerful is facilitated by these individual power study results.

Comparing the individual power results with those of the sequential test performance led to insights as well. In most cases the sequential test power was less than that of its most powerful component test at the same significance level, but almost always greater than the worse of the two. This may seem to favor choosing a single test versus the sequential approach. In many cases though, knowing which test is most powerful might be problematic; hence, the sequential test ensures higher average power against the spectrum of alternatives. In fact there were a couple of cases where both the $\sqrt{b_1}$ and b_2 tests had nearly identical power, and the sequential test power actually exceeded both separate tests at a given significance level. Here, the cooperative nature of the tests overcame the natural lag in power due to the differences in significance level between separate and sequential employment.

A final set of power studies found that incorporating one-sided versions of the skewness and kurtosis tests into the sequential procedure served to increase power quite substantially. The obvious caveat for these modifications is that having enough information on the alternate distributions so that one can identify the proper upper or lower tail test(s) to use is probably rare. Furthermore, introducing one-tailed variants of the two tests changes the attained significance level results, requiring that they be regenerated by more simulation. Nonetheless, if possible, the one-sided tests do improve power of the test in its original form.

5.2 Recommendations for Use

Given the findings from the significance level determination and power studies, several recommendations on the employment of the test can be made:

- When testing for Weibull distributions with $\beta \leq 1$, choose the higher significance level for the skewness test to ensure better power. The nature of the significance levels allows one to select fairly high levels for both tests, particularly at larger sample sizes — pick a combination which allows the highest level for the skewness test and the largest possible corresponding level for the kurtosis test. For these hypotheses, the skewness test tends to be the most powerful; in fact, using just the skewness test as a stand-alone test is a viable option.

- For null hypotheses with $\beta \geq 2$, the first course of action should be to gather some additional information regarding potential alternate distributions for the sample. Using graphical analysis, histograms, or nonparametric density estimates are several effective means of gaining such insights. Theoretical knowledge on the source or process that generated the data may yield other clues. The constant failure rate versus decreasing failure rate example on page 3-32 demonstrates how this information could be derived. If this insight that narrows down the set of alternatives to consider is forthcoming, select the higher significance level for the test that demonstrates the best power for that collection of alternatives. For more or less skewed alternatives, choosing a higher significance level for the $\sqrt{b_1}$ test would be prudent. For cases with symmetric or very similarly skewed alternatives, opting for a combination with a higher significance level for the b_2 test is probably the best course. If specific alternates are known, use the power plots in Appendix D as a basis for identifying the more powerful test statistic. If no such information is available on possible alternates, there are two choices. One can select nearly equal significance levels for both tests, or conduct the sequential test *twice* at the opposite extremes of the possible significance level combinations. For these larger Weibull shapes, the nature of the attained significance levels forces one to chose near equality

at much lower levels of each test or a very biased combination heavily favoring one test over the other. Choosing smaller but nearly equivalent levels will result in lower power for both tests, but gives both tests equal weight. The other option, testing twice at oppositely biased combinations of significance levels (one with the $\sqrt{b_1}$ test at a high level and the other with the b_2 test higher), will give both tests a chance to reject the sample.

- Use the significance level contour plots to help identify the combination of significance levels for the two tests that will enhance power. The more linear the contour levels are, the more advantageous it is to have some additional insights into the possible alternatives one desires to discriminate against since one can bias the combination in favor of one of the tests. These insights might come from theoretical considerations or from graphical analysis of the raw data using histograms or probability plots. If the contours exhibit significant curvature, the decision is much easier because one can usually select a combination in which both tests have a high significance level.
- As with any goodness-of-fit test, conducting it at higher significance levels increases power. This, of course, must be balanced against the desired threshold for Type I error (probability of rejection of the true null hypothesis). Approximating the p-value of the test by conducting it at several larger or smaller significance levels can bolster confidence in the conclusion reached.

5.3 Recommendations for Future Research

The work in this project opens the door for numerous related avenues of further investigation. Several of these are as follows:

- One endeavor that would greatly enhance the utility of this sequential test would be the determination of asymptotic critical values for $\sqrt{b_1}$ and b_2 and a set of functional forms that would permit the approximation of all the critical values as a function of sample size and possibly Weibull shape. This would obviate the extensive array of critical value tables

presented here by Weibull shape and sample size and simplify the application of the test procedure. Such an effort would entail first finding the large-sample critical values for each value of shape and then applying curve-fitting routines to plots of the values by sample size like those presented in Chapter 4. This would express the critical values as a function of the asymptotic values and the sample size n. Taking it one step further, the behavior of the values might be such that not only could a modification by sample size be derived, but also one that would account for shape as well. In the end, perhaps only a single table would be necessary (or one for each Weibull shape), listing critical values by significance level that could be modified with a function of sample size and Weibull shape to determine the values needed for any particular test.

- Given the discussion in Chapter 2 on the ambiguities of identifying just what characteristic of a distribution that kurtosis measures and the fact that the b_2 test statistic demonstrates a high degree of variability (and thus lower utility) for smaller values of shape, it might be fruitful to experiment with alternate test statistics to replace b_2. One such alternative could be Hogg's Q statistic, which he used to replace kurtosis as a discriminator among distributional families for robust parameter estimation [39]. Another option would be to utilize Royston's t_4 statistic, which he developed as a substitute for b_2 [58]. Incorporating one of these test statistics into the kurtosis test, might improve its power and utility in the sequential procedure presented here.

- An even more ambitious goal would be to parallel Bownman and Shenton's work on the distribution of $\sqrt{b_1}$ and b_2 for normal samples which resulted in Johnson S_U system approximations for the joint distribution of the two and omnibus test contours on the $\left(\sqrt{b_1}, b_2\right)$ plane for various significance levels [10]. If a similar work could be done for each Weibull shape, the extensive critical value tables could be eliminated altogether, replaced by a single set of contour plots that would describe the acceptable range of values for both test statistics for a given Weibull shape and sample size.

- Certainly useful as well would be a truly comparable power study using EDF statistics on Wozniak's alternatives based on a three-parameter known shape premise. This would basically entail using Bush's critical values for the EDF statistics for known Weibull shapes and documenting the power for those tests with the same null hypotheses utilized here. The results could then be directly compared with the tables published in this study and more substantative conclusions drawn regarding EDF power versus the sequential test power for these alternatives.

Appendix A. Critical Values

A.1 Sample Skewness Percentage Points

Table A.1 Skewness ($\sqrt{b_1}$) Lower Tail Critical Values: $\beta = 0.5$

Sample	Significance Level (α)									
Size	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
5	-0.748	-0.516	-0.404	-0.359	-0.314	-0.272	-0.230	-0.190	-0.153	-0.118
10	0.068	0.201	0.286	0.344	0.395	0.431	0.466	0.499	0.530	0.558
15	0.422	0.543	0.620	0.675	0.722	0.766	0.805	0.837	0.865	0.891
20	0.674	0.784	0.856	0.913	0.960	0.999	1.031	1.062	1.089	1.115
25	0.852	0.958	1.025	1.078	1.124	1.164	1.199	1.230	1.260	1.288
30	0.979	1.086	1.158	1.217	1.264	1.303	1.340	1.371	1.400	1.427
35	1.112	1.220	1.293	1.343	1.391	1.430	1.465	1.493	1.522	1.548
40	1.200	1.312	1.382	1.434	1.479	1.518	1.555	1.585	1.613	1.641
45	1.300	1.406	1.474	1.527	1.573	1.613	1.650	1.683	1.711	1.737
50	1.374	1.480	1.548	1.603	1.647	1.688	1.722	1.752	1.782	1.807

Sample	Significance Level (α)									
Size	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
5	-0.083	-0.049	-0.017	0.015	0.043	0.068	0.094	0.120	0.144	0.165
10	0.585	0.612	0.634	0.656	0.677	0.699	0.717	0.736	0.753	0.769
15	0.916	0.939	0.963	0.984	1.003	1.024	1.042	1.060	1.077	1.094
20	1.140	1.164	1.185	1.209	1.230	1.249	1.267	1.285	1.304	1.321
25	1.313	1.337	1.361	1.382	1.403	1.422	1.440	1.458	1.476	1.492
30	1.453	1.477	1.498	1.518	1.538	1.557	1.575	1.594	1.612	1.630
35	1.572	1.596	1.620	1.642	1.664	1.683	1.703	1.722	1.739	1.759
40	1.666	1.692	1.715	1.735	1.755	1.776	1.794	1.813	1.831	1.849
45	1.761	1.784	1.805	1.828	1.850	1.871	1.890	1.908	1.926	1.943
50	1.832	1.856	1.878	1.900	1.921	1.941	1.960	1.979	1.997	2.014

Sample	Significance Level (α)									
Size	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
5	0.208	0.244	0.276	0.305	0.331	0.355	0.374	0.390	0.403	0.412
10	0.801	0.831	0.858	0.884	0.908	0.935	0.959	0.983	1.007	1.031
15	1.127	1.156	1.185	1.213	1.239	1.265	1.291	1.315	1.339	1.363
20	1.353	1.383	1.413	1.442	1.469	1.496	1.520	1.546	1.571	1.596
25	1.526	1.558	1.588	1.616	1.644	1.671	1.696	1.721	1.746	1.772
30	1.662	1.695	1.728	1.757	1.785	1.814	1.842	1.867	1.893	1.917
35	1.793	1.823	1.852	1.881	1.911	1.938	1.966	1.993	2.017	2.041
40	1.882	1.914	1.948	1.977	2.006	2.034	2.061	2.087	2.115	2.141
45	1.977	2.008	2.040	2.068	2.096	2.123	2.151	2.177	2.203	2.228
50	2.048	2.081	2.111	2.140	2.170	2.199	2.226	2.252	2.281	2.308

Table A.2 Skewness ($\sqrt{b_1}$) Lower Tail Standard Deviations: $\beta = 0.5$

Sample	Significance Level (α)									
Size	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
5	0.014	0.004	0.003	0.005	0.007	0.008	0.008	0.008	0.007	0.006
10	0.010	0.011	0.009	0.005	0.006	0.004	0.001	0.003	0.002	0.003
15	0.010	0.007	0.009	0.009	0.009	0.006	0.005	0.004	0.005	0.006
20	0.005	0.003	0.003	0.004	0.006	0.007	0.005	0.004	0.005	0.004
25	0.005	0.008	0.002	0.003	0.004	0.003	0.003	0.004	0.003	0.002
30	0.001	0.003	0.003	0.003	0.002	0.003	0.003	0.003	0.001	0.001
35	0.012	0.011	0.013	0.009	0.011	0.010	0.010	0.007	0.008	0.006
40	0.009	0.006	0.007	0.005	0.005	0.005	0.006	0.006	0.004	0.005
45	0.016	0.019	0.013	0.010	0.011	0.011	0.012	0.012	0.012	0.011
50	0.013	0.012	0.010	0.010	0.009	0.010	0.009	0.006	0.008	0.007

Sample	Significance Level (α)									
Size	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
5	0.005	0.004	0.003	0.002	0.003	0.004	0.004	0.004	0.004	0.005
10	0.003	0.002	0.001	0.002	0.002	0.001	0.001	0.001	0.001	0.001
15	0.006	0.005	0.005	0.004	0.004	0.003	0.002	0.002	0.002	0.002
20	0.002	0.003	0.003	0.003	0.003	0.002	0.002	0.002	0.002	0.002
25	0.003	0.004	0.004	0.003	0.002	0.002	0.002	0.002	0.001	0.002
30	0.002	0.001	0.000	0.001	0.001	0.002	0.002	0.001	0.001	0.001
35	0.006	0.006	0.007	0.008	0.008	0.007	0.008	0.008	0.007	0.009
40	0.004	0.005	0.005	0.004	0.003	0.002	0.002	0.002	0.002	0.002
45	0.010	0.009	0.008	0.009	0.010	0.010	0.010	0.010	0.010	0.010
50	0.006	0.006	0.005	0.005	0.005	0.004	0.004	0.003	0.003	0.003

Sample	Significance Level (α)									
Size	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
5	0.003	0.004	0.005	0.004	0.004	0.003	0.003	0.003	0.002	0.002
10	0.001	0.001	0.002	0.003	0.002	0.003	0.003	0.004	0.004	0.003
15	0.002	0.003	0.003	0.004	0.004	0.004	0.003	0.004	0.005	0.004
20	0.002	0.002	0.002	0.002	0.003	0.002	0.002	0.003	0.002	0.002
25	0.002	0.002	0.002	0.002	0.002	0.002	0.001	0.001	0.001	0.000
30	0.002	0.000	0.002	0.002	0.002	0.003	0.003	0.003	0.004	0.003
35	0.007	0.007	0.006	0.006	0.007	0.006	0.006	0.006	0.005	0.004
40	0.001	0.002	0.003	0.004	0.004	0.004	0.003	0.003	0.004	0.005
45	0.009	0.008	0.008	0.006	0.006	0.005	0.004	0.004	0.003	0.002
50	0.003	0.002	0.003	0.002	0.002	0.001	0.002	0.002	0.001	0.000

Table A.3 Skewness ($\sqrt{b_1}$) Upper Tail Critical Values: $\beta = 0.5$

Sample	Significance Level (1-α)									
Size	0.995	0.99	0.985	0.98	0.975	0.97	0.965	0.96	0.955	0.95
5	1.500	1.499	1.499	1.499	1.498	1.497	1.497	1.496	1.495	1.494
10	2.658	2.650	2.643	2.636	2.629	2.622	2.616	2.608	2.600	2.592
15	3.441	3.418	3.398	3.378	3.360	3.341	3.324	3.306	3.288	3.271
20	4.062	4.020	3.983	3.951	3.920	3.890	3.861	3.833	3.805	3.779
25	4.571	4.508	4.457	4.414	4.369	4.327	4.287	4.250	4.212	4.174
30	5.027	4.946	4.874	4.812	4.752	4.697	4.645	4.598	4.551	4.510
35	5.418	5.306	5.221	5.142	5.073	5.010	4.944	4.883	4.828	4.774
40	5.780	5.658	5.554	5.459	5.378	5.298	5.228	5.160	5.097	5.034
45	6.102	5.942	5.817	5.711	5.620	5.539	5.455	5.379	5.305	5.244
50	6.397	6.217	6.073	5.952	5.847	5.745	5.655	5.578	5.494	5.423

Sample	Significance Level (1-α)									
Size	0.945	0.94	0.935	0.93	0.925	0.92	0.915	0.91	0.905	0.9
5	1.493	1.492	1.491	1.490	1.489	1.487	1.486	1.485	1.483	1.481
10	2.584	2.575	2.567	2.559	2.550	2.541	2.532	2.523	2.514	2.505
15	3.254	3.235	3.216	3.198	3.181	3.164	3.149	3.131	3.114	3.096
20	3.751	3.724	3.698	3.672	3.648	3.623	3.598	3.573	3.550	3.524
25	4.141	4.107	4.072	4.036	4.004	3.973	3.942	3.909	3.879	3.849
30	4.463	4.421	4.379	4.341	4.298	4.262	4.225	4.189	4.152	4.118
35	4.723	4.676	4.624	4.579	4.534	4.492	4.449	4.407	4.367	4.329
40	4.978	4.924	4.870	4.817	4.769	4.722	4.675	4.631	4.587	4.543
45	5.181	5.119	5.063	5.008	4.957	4.903	4.853	4.808	4.761	4.714
50	5.353	5.286	5.217	5.158	5.101	5.046	4.991	4.937	4.887	4.838

Sample	Significance Level (1-α)									
Size	0.89	0.88	0.87	0.86	0.85	0.84	0.83	0.82	0.81	0.8
5	1.478	1.475	1.471	1.467	1.462	1.457	1.452	1.446	1.440	1.434
10	2.487	2.467	2.446	2.427	2.405	2.384	2.362	2.341	2.319	2.297
15	3.061	3.028	2.993	2.956	2.922	2.887	2.855	2.822	2.789	2.756
20	3.479	3.429	3.383	3.340	3.297	3.257	3.215	3.173	3.134	3.093
25	3.792	3.735	3.679	3.629	3.579	3.530	3.481	3.432	3.386	3.343
30	4.049	3.985	3.922	3.865	3.807	3.751	3.698	3.647	3.597	3.551
35	4.255	4.185	4.115	4.048	3.986	3.926	3.867	3.815	3.764	3.718
40	4.457	4.381	4.305	4.239	4.172	4.110	4.048	3.988	3.934	3.882
45	4.629	4.548	4.466	4.393	4.323	4.257	4.194	4.136	4.078	4.025
50	4.748	4.662	4.580	4.505	4.436	4.372	4.309	4.249	4.193	4.138

Table A.4 Skewness ($\sqrt{b_1}$) Upper Tail Standard Deviations: $\beta = 0.5$

Sample Size	Significance Level (1-α)									
	0.995	0.99	0.985	0.98	0.975	0.97	0.965	0.96	0.955	0.95
5	0.000	0.000	0.000	0.001	0.000	0.000	0.000	0.000	0.000	0.000
10	0.000	0.001	0.001	0.001	0.001	0.001	0.002	0.001	0.001	0.001
15	0.001	0.002	0.002	0.003	0.003	0.003	0.002	0.003	0.003	0.003
20	0.004	0.002	0.001	0.002	0.003	0.002	0.003	0.003	0.003	0.003
25	0.004	0.006	0.006	0.002	0.003	0.003	0.003	0.004	0.003	0.003
30	0.002	0.003	0.001	0.001	0.002	0.003	0.005	0.005	0.008	0.008
35	0.006	0.009	0.010	0.009	0.008	0.007	0.009	0.012	0.014	0.014
40	0.003	0.004	0.002	0.006	0.009	0.006	0.007	0.007	0.004	0.003
45	0.002	0.012	0.010	0.006	0.003	0.006	0.004	0.006	0.005	0.006
50	0.002	0.006	0.011	0.012	0.008	0.012	0.009	0.009	0.013	0.014

Sample Size	Significance Level (1-α)									
	0.945	0.94	0.935	0.93	0.925	0.92	0.915	0.91	0.905	0.9
5	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.000	0.001	0.001
10	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
15	0.003	0.004	0.005	0.005	0.005	0.004	0.003	0.003	0.004	0.004
20	0.004	0.005	0.004	0.004	0.003	0.002	0.002	0.002	0.002	0.004
25	0.001	0.002	0.003	0.004	0.004	0.004	0.002	0.003	0.004	0.004
30	0.010	0.009	0.008	0.007	0.009	0.007	0.009	0.008	0.012	0.011
35	0.014	0.014	0.016	0.015	0.015	0.013	0.013	0.014	0.014	0.013
40	0.006	0.006	0.006	0.004	0.006	0.006	0.006	0.008	0.008	0.006
45	0.003	0.001	0.004	0.008	0.011	0.009	0.009	0.011	0.012	0.011
50	0.016	0.016	0.017	0.014	0.014	0.012	0.011	0.011	0.012	0.009

Sample Size	Significance Level (1-α)									
	0.89	0.88	0.87	0.86	0.85	0.84	0.83	0.82	0.81	0.8
5	0.001	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.001
10	0.002	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
15	0.005	0.005	0.005	0.006	0.007	0.008	0.006	0.006	0.006	0.007
20	0.004	0.006	0.007	0.006	0.006	0.005	0.004	0.004	0.003	0.004
25	0.005	0.005	0.004	0.003	0.001	0.000	0.002	0.001	0.002	0.003
30	0.011	0.010	0.008	0.008	0.009	0.008	0.008	0.008	0.008	0.007
35	0.013	0.011	0.010	0.011	0.011	0.010	0.011	0.009	0.009	0.007
40	0.004	0.006	0.005	0.006	0.006	0.006	0.004	0.003	0.002	0.002
45	0.012	0.013	0.009	0.009	0.009	0.007	0.007	0.007	0.005	0.005
50	0.007	0.006	0.007	0.006	0.006	0.005	0.004	0.003	0.002	0.003

Table A.5 Skewness ($\sqrt{b_1}$) Lower Tail Critical Values: $\beta = 1.0$

Sample Size	Significance Level (α)									
	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
5	-1.161	-0.988	-0.874	-0.780	-0.704	-0.639	-0.588	-0.541	-0.499	-0.464
10	-0.459	-0.333	-0.250	-0.188	-0.140	-0.096	-0.061	-0.029	-0.003	0.022
15	-0.138	-0.034	0.034	0.082	0.123	0.156	0.188	0.218	0.244	0.266
20	0.051	0.156	0.216	0.268	0.306	0.337	0.365	0.389	0.411	0.431
25	0.191	0.286	0.337	0.380	0.420	0.451	0.479	0.503	0.524	0.543
30	0.301	0.384	0.436	0.477	0.511	0.540	0.564	0.588	0.610	0.628
35	0.398	0.473	0.520	0.561	0.594	0.623	0.648	0.670	0.689	0.707
40	0.452	0.528	0.578	0.616	0.647	0.673	0.698	0.718	0.738	0.756
45	0.511	0.591	0.638	0.674	0.705	0.731	0.755	0.776	0.794	0.811
50	0.568	0.637	0.684	0.719	0.750	0.777	0.796	0.816	0.833	0.850

Sample Size	Significance Level (α)									
	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
5	-0.435	-0.411	-0.392	-0.374	-0.355	-0.337	-0.318	-0.302	-0.284	-0.268
10	0.046	0.069	0.088	0.108	0.126	0.143	0.161	0.178	0.193	0.208
15	0.288	0.307	0.326	0.344	0.360	0.378	0.393	0.409	0.424	0.437
20	0.450	0.470	0.487	0.502	0.519	0.535	0.549	0.563	0.576	0.589
25	0.562	0.579	0.596	0.610	0.625	0.638	0.651	0.664	0.676	0.689
30	0.647	0.663	0.679	0.693	0.708	0.721	0.733	0.746	0.758	0.770
35	0.724	0.739	0.755	0.769	0.783	0.795	0.809	0.820	0.832	0.843
40	0.772	0.788	0.803	0.817	0.831	0.843	0.856	0.867	0.877	0.888
45	0.827	0.842	0.856	0.869	0.883	0.895	0.907	0.917	0.928	0.939
50	0.866	0.881	0.895	0.908	0.921	0.933	0.944	0.955	0.966	0.977

Sample Size	Significance Level (α)									
	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
5	-0.232	-0.200	-0.169	-0.139	-0.108	-0.080	-0.053	-0.025	0.003	0.027
10	0.238	0.265	0.291	0.315	0.338	0.360	0.381	0.402	0.422	0.441
15	0.463	0.488	0.512	0.536	0.557	0.577	0.596	0.615	0.633	0.651
20	0.612	0.635	0.657	0.678	0.699	0.718	0.736	0.756	0.773	0.790
25	0.713	0.735	0.755	0.776	0.796	0.814	0.832	0.850	0.867	0.884
30	0.792	0.813	0.835	0.854	0.873	0.891	0.910	0.927	0.943	0.959
35	0.866	0.886	0.906	0.924	0.942	0.960	0.976	0.992	1.008	1.023
40	0.910	0.930	0.950	0.969	0.986	1.003	1.019	1.035	1.050	1.066
45	0.959	0.977	0.996	1.014	1.031	1.048	1.064	1.081	1.095	1.110
50	0.997	1.016	1.035	1.053	1.070	1.087	1.103	1.119	1.134	1.148

Table A.6 Skewness ($\sqrt{b_1}$) Lower Tail Standard Deviations: $\beta = 1.0$

Sample	Significance Level (α)									
Size	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
5	0.006	0.000	0.007	0.002	0.004	0.003	0.006	0.007	0.005	0.004
10	0.005	0.003	0.005	0.003	0.003	0.005	0.005	0.005	0.003	0.003
15	0.006	0.004	0.003	0.006	0.006	0.007	0.007	0.005	0.006	0.007
20	0.008	0.006	0.006	0.006	0.006	0.005	0.005	0.004	0.005	0.004
25	0.003	0.003	0.005	0.005	0.004	0.004	0.003	0.003	0.003	0.003
30	0.007	0.005	0.003	0.002	0.002	0.002	0.002	0.003	0.003	0.002
35	0.009	0.007	0.003	0.004	0.004	0.006	0.006	0.006	0.006	0.006
40	0.004	0.000	0.003	0.002	0.002	0.002	0.003	0.002	0.001	0.001
45	0.008	0.009	0.007	0.006	0.006	0.005	0.006	0.006	0.005	0.005
50	0.011	0.007	0.005	0.004	0.006	0.006	0.004	0.004	0.003	0.003

Sample	Significance Level (α)									
Size	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
5	0.004	0.004	0.004	0.004	0.004	0.004	0.002	0.004	0.004	0.004
10	0.004	0.003	0.003	0.003	0.003	0.002	0.003	0.002	0.002	0.001
15	0.006	0.006	0.005	0.005	0.005	0.003	0.004	0.003	0.002	0.003
20	0.004	0.002	0.001	0.001	0.000	0.001	0.001	0.001	0.001	0.001
25	0.002	0.001	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002
30	0.003	0.002	0.002	0.001	0.001	0.001	0.002	0.002	0.002	0.002
35	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.005	0.005	0.005
40	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
45	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.003	0.003	0.004
50	0.003	0.003	0.003	0.003	0.003	0.003	0.002	0.002	0.002	0.002

Sample	Significance Level (α)									
Size	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
5	0.003	0.004	0.003	0.003	0.003	0.003	0.004	0.003	0.003	0.004
10	0.003	0.003	0.003	0.002	0.002	0.002	0.002	0.002	0.002	0.002
15	0.003	0.003	0.002	0.001	0.001	0.001	0.001	0.002	0.002	0.002
20	0.000	0.000	0.001	0.001	0.002	0.002	0.001	0.002	0.001	0.002
25	0.002	0.002	0.002	0.001	0.001	0.002	0.001	0.001	0.001	0.001
30	0.002	0.002	0.002	0.002	0.002	0.003	0.003	0.003	0.002	0.002
35	0.006	0.005	0.005	0.005	0.005	0.005	0.004	0.004	0.004	0.003
40	0.001	0.001	0.001	0.001	0.000	0.001	0.001	0.000	0.001	0.000
45	0.003	0.002	0.002	0.003	0.002	0.002	0.002	0.002	0.002	0.003
50	0.003	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.001

Table A.7 Skewness ($\sqrt{b_1}$) Upper Tail Critical Values: $\beta = 1.0$

Sample	Significance Level (1-α)									
Size	0.995	0.99	0.985	0.98	0.975	0.97	0.965	0.96	0.955	0.95
5	1.486	1.477	1.468	1.459	1.450	1.441	1.432	1.424	1.415	1.406
10	2.464	2.389	2.335	2.291	2.249	2.214	2.176	2.143	2.112	2.080
15	2.971	2.834	2.731	2.652	2.586	2.529	2.475	2.427	2.383	2.340
20	3.286	3.097	2.966	2.868	2.785	2.713	2.654	2.600	2.548	2.502
25	3.458	3.241	3.097	2.993	2.905	2.826	2.759	2.698	2.645	2.588
30	3.646	3.379	3.214	3.088	2.985	2.901	2.826	2.762	2.708	2.658
35	3.714	3.425	3.260	3.135	3.030	2.943	2.869	2.801	2.744	2.689
40	3.794	3.510	3.336	3.195	3.082	2.999	2.924	2.857	2.795	2.741
45	3.846	3.526	3.334	3.203	3.095	3.008	2.929	2.865	2.805	2.758
50	3.869	3.558	3.357	3.217	3.114	3.021	2.939	2.869	2.810	2.759

Sample	Significance Level (1-α)									
Size	0.945	0.94	0.935	0.93	0.925	0.92	0.915	0.91	0.905	0.9
5	1.396	1.388	1.379	1.371	1.362	1.353	1.344	1.336	1.326	1.317
10	2.052	2.023	1.998	1.971	1.948	1.924	1.901	1.879	1.857	1.835
15	2.301	2.263	2.228	2.196	2.165	2.137	2.110	2.083	2.058	2.035
20	2.458	2.418	2.382	2.348	2.314	2.286	2.255	2.227	2.199	2.174
25	2.541	2.500	2.461	2.423	2.388	2.358	2.328	2.300	2.274	2.250
30	2.612	2.568	2.530	2.493	2.458	2.425	2.394	2.367	2.340	2.312
35	2.643	2.599	2.564	2.527	2.490	2.456	2.429	2.399	2.371	2.344
40	2.690	2.646	2.604	2.566	2.528	2.495	2.464	2.436	2.408	2.380
45	2.711	2.666	2.625	2.590	2.552	2.523	2.492	2.462	2.434	2.407
50	2.712	2.668	2.626	2.586	2.551	2.520	2.490	2.460	2.435	2.411

Sample	Significance Level (1-α)									
Size	0.89	0.88	0.87	0.86	0.85	0.84	0.83	0.82	0.81	0.8
5	1.299	1.280	1.262	1.241	1.222	1.202	1.183	1.162	1.142	1.123
10	1.795	1.754	1.717	1.680	1.646	1.613	1.584	1.553	1.524	1.495
15	1.991	1.949	1.909	1.872	1.837	1.804	1.772	1.739	1.712	1.686
20	2.125	2.080	2.041	2.001	1.965	1.930	1.900	1.869	1.841	1.813
25	2.201	2.155	2.113	2.075	2.039	2.004	1.971	1.940	1.910	1.882
30	2.263	2.218	2.177	2.137	2.100	2.065	2.032	2.001	1.971	1.942
35	2.296	2.250	2.207	2.168	2.134	2.099	2.067	2.035	2.006	1.980
40	2.333	2.288	2.244	2.205	2.168	2.133	2.101	2.072	2.043	2.014
45	2.358	2.313	2.270	2.231	2.196	2.163	2.131	2.101	2.073	2.045
50	2.365	2.323	2.283	2.246	2.210	2.176	2.146	2.117	2.088	2.061

Table A.8 Skewness ($\sqrt{b_1}$) Upper Tail Standard Deviations: $\beta = 1.0$

Sample Size	Significance Level (1-α)									
	0.995	0.99	0.985	0.98	0.975	0.97	0.965	0.96	0.955	0.95
5	0.001	0.001	0.002	0.002	0.001	0.001	0.002	0.002	0.002	0.002
10	0.002	0.003	0.003	0.000	0.002	0.003	0.002	0.002	0.002	0.001
15	0.009	0.006	0.009	0.011	0.009	0.009	0.007	0.008	0.007	0.008
20	0.006	0.009	0.011	0.006	0.006	0.007	0.005	0.003	0.002	0.001
25	0.025	0.019	0.014	0.009	0.004	0.005	0.005	0.004	0.003	0.008
30	0.012	0.009	0.005	0.004	0.008	0.006	0.005	0.005	0.004	0.004
35	0.016	0.017	0.010	0.002	0.002	0.000	0.003	0.003	0.003	0.004
40	0.013	0.006	0.009	0.008	0.007	0.010	0.014	0.012	0.010	0.010
45	0.015	0.022	0.020	0.010	0.009	0.003	0.005	0.002	0.001	0.002
50	0.008	0.006	0.008	0.007	0.006	0.005	0.004	0.006	0.006	0.004

Sample Size	Significance Level (1-α)									
	0.945	0.94	0.935	0.93	0.925	0.92	0.915	0.91	0.905	0.9
5	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
10	0.002	0.003	0.004	0.004	0.005	0.005	0.005	0.006	0.007	0.007
15	0.009	0.009	0.010	0.008	0.008	0.007	0.007	0.007	0.006	0.005
20	0.003	0.003	0.004	0.004	0.004	0.005	0.004	0.003	0.002	0.003
25	0.010	0.008	0.008	0.009	0.009	0.008	0.006	0.006	0.006	0.005
30	0.006	0.006	0.007	0.005	0.006	0.004	0.003	0.003	0.004	0.003
35	0.003	0.003	0.001	0.003	0.004	0.004	0.002	0.002	0.002	0.003
40	0.008	0.006	0.006	0.007	0.006	0.006	0.005	0.007	0.006	0.007
45	0.003	0.002	0.002	0.003	0.002	0.004	0.003	0.004	0.003	0.004
50	0.004	0.004	0.005	0.007	0.007	0.006	0.006	0.006	0.005	0.003

Sample Size	Significance Level (1-α)									
	0.89	0.88	0.87	0.86	0.85	0.84	0.83	0.82	0.81	0.8
5	0.002	0.001	0.002	0.001	0.001	0.001	0.001	0.002	0.002	0.002
10	0.006	0.005	0.006	0.005	0.005	0.005	0.006	0.004	0.003	0.003
15	0.003	0.002	0.002	0.003	0.002	0.003	0.003	0.005	0.005	0.004
20	0.002	0.002	0.002	0.001	0.001	0.001	0.002	0.003	0.003	0.003
25	0.005	0.005	0.005	0.005	0.004	0.005	0.005	0.006	0.005	0.005
30	0.003	0.004	0.004	0.004	0.004	0.003	0.002	0.003	0.003	0.003
35	0.003	0.004	0.005	0.004	0.003	0.004	0.004	0.004	0.004	0.002
40	0.007	0.007	0.006	0.006	0.006	0.005	0.005	0.006	0.005	0.004
45	0.004	0.004	0.002	0.003	0.004	0.004	0.004	0.004	0.004	0.004
50	0.002	0.001	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.002

Table A.9 Skewness ($\sqrt{b_1}$) Lower Tail Critical Values: $\beta = 1.5$

Sample Size	Significance Level (α)									
	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
5	-1.282	-1.163	-1.070	-0.998	-0.936	-0.879	-0.827	-0.785	-0.744	-0.702
10	-0.801	-0.653	-0.570	-0.511	-0.458	-0.418	-0.381	-0.348	-0.318	-0.292
15	-0.486	-0.376	-0.309	-0.261	-0.223	-0.189	-0.158	-0.130	-0.106	-0.084
20	-0.310	-0.209	-0.146	-0.103	-0.066	-0.036	-0.008	0.016	0.038	0.056
25	-0.182	-0.095	-0.043	-0.002	0.031	0.059	0.085	0.106	0.125	0.142
30	-0.082	-0.005	0.044	0.078	0.107	0.133	0.153	0.172	0.193	0.210
35	-0.003	0.071	0.111	0.147	0.176	0.200	0.222	0.241	0.259	0.275
40	0.043	0.114	0.159	0.192	0.218	0.241	0.262	0.279	0.295	0.312
45	0.099	0.165	0.208	0.239	0.265	0.286	0.305	0.322	0.338	0.352
50	0.138	0.204	0.243	0.273	0.300	0.320	0.340	0.356	0.372	0.384

Sample Size	Significance Level (α)									
	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
5	-0.667	-0.633	-0.603	-0.575	-0.549	-0.525	-0.503	-0.483	-0.463	-0.446
10	-0.271	-0.248	-0.227	-0.208	-0.190	-0.172	-0.155	-0.139	-0.124	-0.109
15	-0.064	-0.045	-0.027	-0.010	0.005	0.020	0.034	0.047	0.060	0.073
20	0.074	0.090	0.106	0.120	0.134	0.147	0.160	0.172	0.184	0.196
25	0.159	0.174	0.189	0.202	0.216	0.228	0.240	0.251	0.262	0.274
30	0.226	0.241	0.255	0.268	0.280	0.292	0.303	0.314	0.324	0.333
35	0.289	0.303	0.315	0.327	0.338	0.349	0.359	0.369	0.379	0.389
40	0.325	0.337	0.349	0.360	0.371	0.382	0.392	0.402	0.412	0.421
45	0.365	0.378	0.390	0.401	0.411	0.422	0.431	0.440	0.449	0.457
50	0.396	0.409	0.420	0.431	0.442	0.452	0.460	0.469	0.479	0.487

Sample Size	Significance Level (α)									
	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
5	-0.414	-0.389	-0.363	-0.338	-0.314	-0.292	-0.269	-0.246	-0.223	-0.199
10	-0.081	-0.057	-0.031	-0.006	0.016	0.038	0.058	0.077	0.096	0.115
15	0.098	0.122	0.143	0.164	0.184	0.203	0.221	0.238	0.254	0.272
20	0.217	0.237	0.256	0.275	0.292	0.309	0.326	0.341	0.357	0.372
25	0.295	0.314	0.332	0.349	0.366	0.381	0.397	0.411	0.425	0.439
30	0.353	0.371	0.388	0.404	0.420	0.436	0.450	0.464	0.478	0.491
35	0.406	0.424	0.439	0.454	0.469	0.482	0.495	0.509	0.522	0.535
40	0.437	0.454	0.469	0.484	0.499	0.513	0.526	0.539	0.552	0.564
45	0.473	0.489	0.503	0.517	0.531	0.544	0.557	0.570	0.582	0.594
50	0.503	0.518	0.532	0.546	0.560	0.572	0.584	0.596	0.608	0.619

Table A.10 Skewness ($\sqrt{b_1}$) Upper Tail Critical Values: $\beta = 1.5$

Sample Size	Significance Level (1-α)									
	0.995	0.99	0.985	0.98	0.975	0.97	0.965	0.96	0.955	0.95
5	1.458	1.434	1.412	1.392	1.373	1.355	1.338	1.320	1.301	1.285
10	2.152	2.025	1.940	1.874	1.816	1.765	1.718	1.678	1.641	1.607
15	2.357	2.184	2.064	1.976	1.909	1.846	1.795	1.751	1.710	1.672
20	2.445	2.229	2.103	2.016	1.943	1.886	1.835	1.790	1.747	1.710
25	2.422	2.227	2.094	2.009	1.935	1.880	1.832	1.785	1.748	1.709
30	2.448	2.238	2.101	2.007	1.936	1.877	1.828	1.784	1.746	1.711
35	2.416	2.205	2.079	1.989	1.917	1.859	1.808	1.764	1.726	1.696
40	2.384	2.181	2.058	1.970	1.905	1.852	1.803	1.760	1.722	1.691
45	2.363	2.160	2.035	1.947	1.880	1.825	1.781	1.739	1.706	1.674
50	2.338	2.131	2.007	1.924	1.861	1.809	1.760	1.721	1.687	1.656

Sample Size	Significance Level (1-α)									
	0.945	0.94	0.935	0.93	0.925	0.92	0.915	0.91	0.905	0.9
5	1.268	1.253	1.238	1.221	1.206	1.192	1.177	1.163	1.149	1.135
10	1.575	1.542	1.512	1.487	1.462	1.438	1.415	1.393	1.372	1.354
15	1.640	1.608	1.578	1.552	1.527	1.504	1.481	1.460	1.438	1.420
20	1.676	1.647	1.616	1.591	1.566	1.542	1.521	1.498	1.479	1.460
25	1.675	1.646	1.617	1.590	1.566	1.544	1.522	1.501	1.480	1.461
30	1.679	1.648	1.622	1.597	1.574	1.550	1.530	1.511	1.492	1.474
35	1.666	1.638	1.612	1.587	1.564	1.543	1.522	1.503	1.486	1.468
40	1.663	1.638	1.611	1.586	1.564	1.543	1.524	1.505	1.487	1.470
45	1.645	1.618	1.597	1.575	1.555	1.535	1.517	1.498	1.482	1.466
50	1.629	1.603	1.580	1.558	1.538	1.520	1.502	1.486	1.470	1.454

Sample Size	Significance Level (1-α)									
	0.89	0.88	0.87	0.86	0.85	0.84	0.83	0.82	0.81	0.8
5	1.108	1.080	1.054	1.026	0.998	0.973	0.948	0.923	0.899	0.873
10	1.315	1.279	1.246	1.216	1.186	1.159	1.133	1.108	1.084	1.060
15	1.382	1.348	1.317	1.287	1.259	1.233	1.209	1.185	1.161	1.139
20	1.424	1.391	1.359	1.329	1.301	1.274	1.251	1.228	1.207	1.186
25	1.426	1.394	1.365	1.337	1.311	1.287	1.265	1.243	1.223	1.203
30	1.441	1.411	1.381	1.353	1.328	1.304	1.282	1.260	1.241	1.222
35	1.435	1.408	1.379	1.354	1.330	1.308	1.286	1.266	1.246	1.227
40	1.438	1.408	1.382	1.356	1.333	1.311	1.290	1.271	1.252	1.235
45	1.436	1.407	1.381	1.356	1.333	1.313	1.292	1.274	1.256	1.239
50	1.425	1.399	1.375	1.352	1.332	1.312	1.293	1.276	1.259	1.243

Table A.11 Skewness ($\sqrt{b_1}$) Lower Tail Critical Values: $\beta = 2.0$

Sample	Significance Level (α)									
Size	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
5	-1.334	-1.242	-1.169	-1.108	-1.053	-1.006	-0.960	-0.920	-0.882	-0.844
10	-1.041	-0.890	-0.796	-0.732	-0.680	-0.639	-0.602	-0.567	-0.538	-0.510
15	-0.739	-0.624	-0.555	-0.502	-0.463	-0.427	-0.396	-0.371	-0.348	-0.326
20	-0.568	-0.465	-0.403	-0.358	-0.322	-0.292	-0.267	-0.243	-0.221	-0.201
25	-0.447	-0.357	-0.307	-0.266	-0.234	-0.206	-0.180	-0.160	-0.141	-0.123
30	-0.355	-0.275	-0.227	-0.191	-0.164	-0.138	-0.115	-0.096	-0.078	-0.063
35	-0.278	-0.205	-0.161	-0.129	-0.098	-0.076	-0.057	-0.038	-0.023	-0.008
40	-0.232	-0.163	-0.121	-0.089	-0.065	-0.044	-0.024	-0.007	0.008	0.023
45	-0.182	-0.117	-0.076	-0.045	-0.021	-0.001	0.016	0.032	0.047	0.060
50	-0.142	-0.083	-0.043	-0.014	0.011	0.030	0.046	0.063	0.077	0.090

Sample	Significance Level (α)									
Size	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
5	-0.811	-0.780	-0.751	-0.724	-0.695	-0.671	-0.646	-0.625	-0.602	-0.583
10	-0.486	-0.463	-0.443	-0.423	-0.403	-0.385	-0.369	-0.353	-0.337	-0.321
15	-0.305	-0.285	-0.267	-0.252	-0.235	-0.220	-0.206	-0.192	-0.179	-0.167
20	-0.183	-0.166	-0.151	-0.137	-0.122	-0.110	-0.098	-0.086	-0.074	-0.063
25	-0.106	-0.091	-0.076	-0.063	-0.049	-0.037	-0.026	-0.015	-0.005	0.005
30	-0.048	-0.034	-0.021	-0.009	0.002	0.014	0.025	0.035	0.044	0.055
35	0.006	0.019	0.031	0.042	0.052	0.063	0.073	0.083	0.091	0.100
40	0.037	0.048	0.060	0.071	0.082	0.092	0.102	0.111	0.119	0.128
45	0.072	0.084	0.094	0.105	0.115	0.125	0.134	0.143	0.151	0.159
50	0.103	0.113	0.123	0.133	0.143	0.151	0.160	0.169	0.176	0.183

Sample	Significance Level (α)									
Size	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
5	-0.546	-0.511	-0.481	-0.453	-0.427	-0.404	-0.384	-0.364	-0.344	-0.325
10	-0.293	-0.266	-0.242	-0.218	-0.195	-0.173	-0.153	-0.132	-0.113	-0.096
15	-0.143	-0.120	-0.099	-0.080	-0.060	-0.041	-0.023	-0.006	0.010	0.026
20	-0.043	-0.023	-0.004	0.014	0.031	0.047	0.062	0.078	0.092	0.106
25	0.024	0.041	0.058	0.074	0.090	0.104	0.119	0.133	0.147	0.159
30	0.073	0.089	0.106	0.121	0.136	0.150	0.163	0.176	0.189	0.201
35	0.117	0.132	0.147	0.161	0.174	0.186	0.199	0.211	0.223	0.235
40	0.144	0.159	0.173	0.187	0.199	0.212	0.224	0.235	0.246	0.257
45	0.174	0.188	0.202	0.215	0.227	0.239	0.250	0.260	0.270	0.281
50	0.198	0.212	0.224	0.237	0.248	0.259	0.270	0.281	0.292	0.301

Table A.12 Skewness ($\sqrt{b_1}$) Upper Tail Critical Values: $\beta = 2.0$

Sample Size	Significance Level (1-α)									
	0.995	0.99	0.985	0.98	0.975	0.97	0.965	0.96	0.955	0.95
5	1.434	1.398	1.365	1.335	1.307	1.282	1.258	1.236	1.214	1.190
10	1.896	1.748	1.651	1.574	1.509	1.457	1.412	1.371	1.335	1.302
15	1.921	1.742	1.633	1.554	1.492	1.442	1.397	1.357	1.321	1.289
20	1.881	1.702	1.600	1.520	1.464	1.415	1.373	1.335	1.303	1.271
25	1.810	1.644	1.541	1.470	1.417	1.368	1.330	1.295	1.263	1.234
30	1.766	1.614	1.518	1.441	1.387	1.340	1.304	1.269	1.239	1.213
35	1.731	1.570	1.475	1.405	1.349	1.309	1.271	1.238	1.210	1.188
40	1.668	1.522	1.427	1.364	1.317	1.279	1.245	1.215	1.190	1.166
45	1.631	1.482	1.398	1.339	1.292	1.254	1.221	1.192	1.168	1.147
50	1.592	1.452	1.368	1.310	1.266	1.230	1.197	1.171	1.147	1.124

Sample Size	Significance Level (1-α)									
	0.945	0.94	0.935	0.93	0.925	0.92	0.915	0.91	0.905	0.9
5	1.169	1.148	1.128	1.110	1.091	1.073	1.054	1.037	1.019	1.003
10	1.271	1.243	1.218	1.195	1.173	1.151	1.130	1.108	1.089	1.071
15	1.259	1.232	1.205	1.181	1.159	1.138	1.118	1.099	1.082	1.064
20	1.244	1.219	1.195	1.171	1.148	1.129	1.111	1.093	1.076	1.061
25	1.208	1.184	1.163	1.142	1.122	1.104	1.086	1.070	1.053	1.038
30	1.190	1.167	1.146	1.128	1.110	1.092	1.076	1.060	1.045	1.031
35	1.163	1.143	1.124	1.105	1.087	1.071	1.056	1.042	1.028	1.014
40	1.145	1.125	1.106	1.088	1.073	1.059	1.044	1.030	1.016	1.003
45	1.126	1.107	1.090	1.073	1.057	1.041	1.028	1.014	1.002	0.990
50	1.104	1.085	1.068	1.052	1.038	1.024	1.011	0.999	0.987	0.976

Sample Size	Significance Level (1-α)									
	0.89	0.88	0.87	0.86	0.85	0.84	0.83	0.82	0.81	0.8
5	0.971	0.939	0.907	0.879	0.851	0.821	0.793	0.767	0.740	0.715
10	1.035	1.002	0.971	0.944	0.915	0.891	0.866	0.844	0.821	0.799
15	1.033	1.004	0.975	0.950	0.927	0.902	0.880	0.860	0.839	0.820
20	1.031	1.002	0.977	0.953	0.930	0.909	0.889	0.869	0.851	0.834
25	1.010	0.984	0.959	0.937	0.916	0.895	0.877	0.860	0.842	0.826
30	1.003	0.978	0.956	0.936	0.916	0.898	0.880	0.864	0.846	0.830
35	0.988	0.966	0.945	0.924	0.906	0.887	0.870	0.854	0.840	0.825
40	0.980	0.958	0.938	0.919	0.901	0.883	0.868	0.852	0.837	0.822
45	0.967	0.946	0.926	0.908	0.891	0.875	0.861	0.846	0.832	0.819
50	0.954	0.935	0.917	0.900	0.885	0.870	0.855	0.841	0.827	0.815

Table A.13 Skewness ($\sqrt{b_1}$) Lower Tail Critical Values: $\beta = 2.5$

Sample Size	Significance Level (α)									
	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
5	-1.363	-1.285	-1.223	-1.171	-1.122	-1.080	-1.039	-1.002	-0.967	-0.934
10	-1.222	-1.066	-0.968	-0.901	-0.850	-0.805	-0.766	-0.729	-0.700	-0.673
15	-0.941	-0.819	-0.749	-0.693	-0.649	-0.612	-0.579	-0.550	-0.527	-0.505
20	-0.773	-0.670	-0.603	-0.559	-0.520	-0.487	-0.460	-0.436	-0.414	-0.394
25	-0.659	-0.566	-0.506	-0.465	-0.431	-0.402	-0.379	-0.357	-0.338	-0.321
30	-0.567	-0.481	-0.430	-0.395	-0.364	-0.339	-0.318	-0.298	-0.281	-0.266
35	-0.490	-0.415	-0.370	-0.334	-0.308	-0.286	-0.264	-0.245	-0.228	-0.214
40	-0.441	-0.374	-0.331	-0.299	-0.274	-0.251	-0.232	-0.216	-0.200	-0.186
45	-0.394	-0.328	-0.286	-0.257	-0.232	-0.212	-0.193	-0.178	-0.164	-0.152
50	-0.356	-0.294	-0.255	-0.225	-0.203	-0.182	-0.166	-0.151	-0.136	-0.123

Sample Size	Significance Level (α)									
	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
5	-0.903	-0.872	-0.846	-0.818	-0.793	-0.768	-0.746	-0.724	-0.703	-0.682
10	-0.646	-0.621	-0.599	-0.580	-0.560	-0.541	-0.523	-0.505	-0.490	-0.475
15	-0.485	-0.466	-0.446	-0.428	-0.412	-0.397	-0.382	-0.368	-0.355	-0.342
20	-0.374	-0.358	-0.342	-0.327	-0.313	-0.300	-0.286	-0.274	-0.263	-0.251
25	-0.304	-0.288	-0.274	-0.260	-0.246	-0.234	-0.221	-0.210	-0.200	-0.191
30	-0.250	-0.236	-0.223	-0.211	-0.198	-0.185	-0.175	-0.165	-0.155	-0.146
35	-0.200	-0.187	-0.176	-0.165	-0.154	-0.144	-0.134	-0.125	-0.116	-0.107
40	-0.172	-0.160	-0.148	-0.137	-0.127	-0.117	-0.107	-0.098	-0.090	-0.082
45	-0.140	-0.127	-0.116	-0.106	-0.095	-0.086	-0.077	-0.068	-0.060	-0.053
50	-0.111	-0.100	-0.090	-0.080	-0.071	-0.063	-0.054	-0.046	-0.039	-0.031

Sample Size	Significance Level (α)									
	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
5	-0.644	-0.609	-0.575	-0.544	-0.516	-0.489	-0.463	-0.440	-0.419	-0.401
10	-0.446	-0.418	-0.391	-0.368	-0.345	-0.325	-0.304	-0.284	-0.264	-0.245
15	-0.319	-0.295	-0.274	-0.253	-0.234	-0.216	-0.197	-0.180	-0.164	-0.148
20	-0.231	-0.211	-0.193	-0.175	-0.157	-0.141	-0.125	-0.110	-0.094	-0.081
25	-0.171	-0.153	-0.137	-0.122	-0.107	-0.092	-0.078	-0.064	-0.051	-0.037
30	-0.129	-0.112	-0.096	-0.081	-0.066	-0.053	-0.039	-0.027	-0.015	-0.003
35	-0.091	-0.075	-0.060	-0.046	-0.034	-0.021	-0.009	0.002	0.013	0.025
40	-0.066	-0.052	-0.037	-0.024	-0.011	0.001	0.013	0.024	0.035	0.045
45	-0.038	-0.024	-0.011	0.001	0.013	0.025	0.035	0.046	0.056	0.066
50	-0.017	-0.004	0.009	0.020	0.031	0.042	0.053	0.063	0.073	0.083

Table A.14 Skewness ($\sqrt{b_1}$) Upper Tail Critical Values: $\beta = 2.5$

Sample Size	Significance Level (1-α)									
	0.995	0.99	0.985	0.98	0.975	0.97	0.965	0.96	0.955	0.95
5	1.415	1.369	1.330	1.292	1.261	1.228	1.200	1.173	1.147	1.122
10	1.716	1.558	1.454	1.376	1.316	1.265	1.220	1.182	1.144	1.116
15	1.645	1.469	1.374	1.300	1.245	1.193	1.147	1.111	1.079	1.050
20	1.539	1.387	1.294	1.227	1.176	1.133	1.096	1.061	1.031	1.005
25	1.448	1.300	1.218	1.158	1.109	1.070	1.034	1.002	0.974	0.951
30	1.389	1.257	1.174	1.115	1.067	1.031	0.998	0.969	0.943	0.920
35	1.337	1.204	1.129	1.070	1.026	0.991	0.961	0.934	0.910	0.889
40	1.267	1.149	1.078	1.023	0.986	0.954	0.927	0.902	0.880	0.861
45	1.218	1.113	1.042	0.996	0.961	0.931	0.904	0.882	0.859	0.840
50	1.189	1.080	1.015	0.969	0.930	0.900	0.874	0.853	0.833	0.815

Sample Size	Significance Level (1-α)									
	0.945	0.94	0.935	0.93	0.925	0.92	0.915	0.91	0.905	0.9
5	1.098	1.075	1.052	1.031	1.011	0.989	0.969	0.949	0.930	0.912
10	1.087	1.058	1.032	1.010	0.988	0.966	0.947	0.928	0.909	0.891
15	1.024	0.998	0.975	0.954	0.934	0.915	0.897	0.879	0.863	0.847
20	0.982	0.958	0.935	0.916	0.898	0.879	0.863	0.846	0.832	0.817
25	0.929	0.909	0.890	0.872	0.855	0.839	0.823	0.808	0.793	0.779
30	0.900	0.881	0.863	0.846	0.830	0.815	0.801	0.788	0.775	0.762
35	0.869	0.852	0.835	0.819	0.805	0.790	0.776	0.763	0.751	0.739
40	0.842	0.826	0.810	0.796	0.782	0.769	0.757	0.745	0.734	0.724
45	0.822	0.805	0.790	0.776	0.764	0.751	0.740	0.727	0.716	0.706
50	0.799	0.785	0.770	0.757	0.744	0.733	0.720	0.710	0.701	0.692

Sample Size	Significance Level (1-α)									
	0.89	0.88	0.87	0.86	0.85	0.84	0.83	0.82	0.81	0.8
5	0.877	0.843	0.811	0.781	0.751	0.721	0.693	0.666	0.638	0.613
10	0.857	0.826	0.799	0.772	0.747	0.723	0.698	0.676	0.655	0.635
15	0.817	0.790	0.763	0.737	0.716	0.695	0.675	0.655	0.637	0.620
20	0.791	0.766	0.742	0.720	0.699	0.680	0.662	0.644	0.627	0.611
25	0.754	0.731	0.711	0.690	0.672	0.655	0.638	0.623	0.607	0.592
30	0.740	0.719	0.699	0.681	0.663	0.646	0.630	0.615	0.600	0.586
35	0.718	0.698	0.680	0.664	0.648	0.631	0.616	0.602	0.588	0.575
40	0.704	0.685	0.668	0.651	0.635	0.620	0.606	0.593	0.580	0.567
45	0.686	0.669	0.653	0.637	0.622	0.608	0.595	0.582	0.571	0.559
50	0.674	0.657	0.642	0.627	0.613	0.600	0.587	0.574	0.563	0.552

Table A.15 Skewness ($\sqrt{b_1}$) Lower Tail Critical Values: $\beta = 3.0$

Sample Size	Significance Level (α)									
	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
5	-1.382	-1.312	-1.256	-1.213	-1.167	-1.128	-1.090	-1.057	-1.023	-0.993
10	-1.368	-1.201	-1.106	-1.035	-0.980	-0.930	-0.891	-0.856	-0.824	-0.792
15	-1.108	-0.984	-0.903	-0.842	-0.797	-0.761	-0.727	-0.698	-0.669	-0.647
20	-0.952	-0.840	-0.769	-0.721	-0.679	-0.646	-0.615	-0.588	-0.565	-0.544
25	-0.835	-0.736	-0.670	-0.626	-0.592	-0.562	-0.537	-0.514	-0.494	-0.476
30	-0.739	-0.653	-0.599	-0.559	-0.527	-0.502	-0.479	-0.459	-0.441	-0.423
35	-0.670	-0.587	-0.539	-0.503	-0.473	-0.449	-0.427	-0.408	-0.393	-0.378
40	-0.615	-0.541	-0.498	-0.465	-0.439	-0.418	-0.398	-0.380	-0.363	-0.348
45	-0.568	-0.499	-0.455	-0.423	-0.400	-0.379	-0.360	-0.343	-0.328	-0.315
50	-0.527	-0.464	-0.424	-0.395	-0.372	-0.352	-0.332	-0.316	-0.302	-0.288

Sample Size	Significance Level (α)									
	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
5	-0.964	-0.937	-0.910	-0.885	-0.860	-0.837	-0.815	-0.794	-0.773	-0.752
10	-0.767	-0.743	-0.722	-0.700	-0.679	-0.661	-0.642	-0.625	-0.609	-0.592
15	-0.625	-0.605	-0.586	-0.567	-0.551	-0.535	-0.520	-0.505	-0.492	-0.478
20	-0.525	-0.507	-0.491	-0.475	-0.461	-0.448	-0.436	-0.422	-0.410	-0.399
25	-0.458	-0.442	-0.428	-0.415	-0.401	-0.389	-0.377	-0.365	-0.354	-0.343
30	-0.407	-0.394	-0.381	-0.368	-0.357	-0.344	-0.333	-0.322	-0.311	-0.301
35	-0.363	-0.351	-0.338	-0.326	-0.315	-0.304	-0.295	-0.285	-0.276	-0.266
40	-0.334	-0.321	-0.309	-0.298	-0.288	-0.278	-0.269	-0.260	-0.251	-0.243
45	-0.303	-0.291	-0.279	-0.269	-0.259	-0.248	-0.240	-0.232	-0.223	-0.215
50	-0.276	-0.264	-0.254	-0.245	-0.236	-0.227	-0.218	-0.211	-0.203	-0.195

Sample Size	Significance Level (α)									
	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
5	-0.713	-0.678	-0.645	-0.615	-0.585	-0.556	-0.529	-0.503	-0.481	-0.458
10	-0.561	-0.532	-0.507	-0.482	-0.458	-0.435	-0.414	-0.393	-0.375	-0.355
15	-0.453	-0.429	-0.406	-0.386	-0.366	-0.346	-0.328	-0.310	-0.294	-0.279
20	-0.376	-0.355	-0.336	-0.318	-0.301	-0.284	-0.267	-0.251	-0.236	-0.222
25	-0.324	-0.305	-0.287	-0.270	-0.255	-0.241	-0.226	-0.212	-0.199	-0.185
30	-0.283	-0.266	-0.250	-0.235	-0.221	-0.207	-0.193	-0.180	-0.168	-0.155
35	-0.249	-0.234	-0.219	-0.205	-0.191	-0.179	-0.166	-0.154	-0.143	-0.132
40	-0.227	-0.212	-0.197	-0.183	-0.171	-0.159	-0.146	-0.135	-0.125	-0.114
45	-0.201	-0.186	-0.173	-0.161	-0.149	-0.138	-0.127	-0.116	-0.106	-0.096
50	-0.181	-0.168	-0.155	-0.143	-0.132	-0.121	-0.111	-0.100	-0.090	-0.081

Table A.16 Skewness ($\sqrt{b_1}$) Upper Tail Critical Values: $\beta = 3.0$

Sample	Significance Level (1-α)									
Size	0.995	0.99	0.985	0.98	0.975	0.97	0.965	0.96	0.955	0.95
5	1.401	1.350	1.303	1.261	1.225	1.191	1.158	1.128	1.100	1.073
10	1.591	1.428	1.322	1.250	1.188	1.138	1.095	1.057	1.022	0.993
15	1.468	1.293	1.205	1.137	1.076	1.031	0.988	0.954	0.923	0.895
20	1.330	1.183	1.101	1.038	0.993	0.952	0.916	0.885	0.858	0.832
25	1.223	1.093	1.016	0.957	0.913	0.876	0.846	0.817	0.793	0.770
30	1.151	1.036	0.960	0.910	0.869	0.832	0.803	0.776	0.753	0.731
35	1.092	0.977	0.910	0.860	0.823	0.790	0.761	0.738	0.717	0.698
40	1.024	0.920	0.857	0.816	0.782	0.751	0.724	0.702	0.681	0.663
45	0.973	0.877	0.821	0.783	0.750	0.726	0.702	0.681	0.660	0.643
50	0.949	0.850	0.794	0.753	0.721	0.693	0.669	0.649	0.632	0.616

Sample	Significance Level (1-α)									
Size	0.945	0.94	0.935	0.93	0.925	0.92	0.915	0.91	0.905	0.9
5	1.046	1.021	0.997	0.975	0.951	0.931	0.910	0.888	0.868	0.849
10	0.963	0.937	0.911	0.888	0.864	0.843	0.824	0.806	0.789	0.772
15	0.868	0.847	0.823	0.803	0.785	0.767	0.749	0.732	0.716	0.701
20	0.809	0.787	0.768	0.749	0.731	0.716	0.699	0.684	0.671	0.657
25	0.748	0.728	0.710	0.694	0.679	0.663	0.649	0.635	0.622	0.610
30	0.712	0.695	0.678	0.663	0.649	0.635	0.622	0.610	0.598	0.587
35	0.680	0.662	0.645	0.631	0.617	0.605	0.592	0.581	0.570	0.560
40	0.648	0.632	0.618	0.606	0.594	0.582	0.571	0.560	0.550	0.540
45	0.627	0.611	0.598	0.584	0.573	0.561	0.550	0.539	0.529	0.520
50	0.601	0.588	0.576	0.564	0.553	0.542	0.532	0.522	0.513	0.504

Sample	Significance Level (1-α)									
Size	0.89	0.88	0.87	0.86	0.85	0.84	0.83	0.82	0.81	0.8
5	0.812	0.777	0.744	0.713	0.682	0.653	0.623	0.597	0.573	0.547
10	0.740	0.709	0.681	0.656	0.631	0.608	0.585	0.564	0.543	0.523
15	0.673	0.646	0.622	0.599	0.577	0.556	0.537	0.518	0.500	0.482
20	0.632	0.608	0.586	0.565	0.544	0.526	0.508	0.492	0.476	0.460
25	0.587	0.565	0.545	0.527	0.509	0.494	0.478	0.463	0.447	0.433
30	0.564	0.545	0.527	0.509	0.493	0.477	0.463	0.449	0.435	0.421
35	0.541	0.523	0.505	0.488	0.473	0.458	0.443	0.430	0.418	0.405
40	0.521	0.503	0.487	0.472	0.457	0.443	0.429	0.417	0.405	0.393
45	0.502	0.487	0.471	0.456	0.442	0.429	0.417	0.405	0.394	0.383
50	0.487	0.471	0.457	0.443	0.431	0.419	0.407	0.396	0.385	0.375

Table A.17 Skewness ($\sqrt{b_1}$) Lower Tail Critical Values: $\beta = 3.5$

Sample	Significance Level (α)									
Size	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
5	-1.394	-1.332	-1.280	-1.239	-1.198	-1.161	-1.128	-1.095	-1.063	-1.035
10	-1.481	-1.318	-1.214	-1.140	-1.081	-1.034	-0.993	-0.955	-0.920	-0.891
15	-1.255	-1.108	-1.029	-0.970	-0.921	-0.882	-0.848	-0.818	-0.789	-0.764
20	-1.106	-0.984	-0.909	-0.850	-0.810	-0.774	-0.744	-0.714	-0.690	-0.668
25	-0.986	-0.879	-0.810	-0.762	-0.727	-0.696	-0.667	-0.643	-0.622	-0.603
30	-0.887	-0.799	-0.738	-0.698	-0.665	-0.635	-0.612	-0.591	-0.572	-0.555
35	-0.818	-0.735	-0.683	-0.644	-0.611	-0.586	-0.565	-0.546	-0.528	-0.511
40	-0.763	-0.685	-0.639	-0.605	-0.577	-0.553	-0.534	-0.515	-0.497	-0.480
45	-0.721	-0.642	-0.597	-0.564	-0.539	-0.517	-0.498	-0.480	-0.463	-0.449
50	-0.674	-0.609	-0.567	-0.535	-0.511	-0.489	-0.470	-0.452	-0.437	-0.423

Sample	Significance Level (α)									
Size	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
5	-1.008	-0.982	-0.957	-0.932	-0.909	-0.886	-0.866	-0.844	-0.825	-0.803
10	-0.864	-0.840	-0.817	-0.795	-0.774	-0.754	-0.734	-0.716	-0.699	-0.682
15	-0.739	-0.718	-0.698	-0.679	-0.661	-0.644	-0.629	-0.614	-0.599	-0.585
20	-0.649	-0.631	-0.613	-0.597	-0.582	-0.566	-0.553	-0.541	-0.528	-0.516
25	-0.586	-0.569	-0.554	-0.539	-0.526	-0.512	-0.500	-0.488	-0.476	-0.466
30	-0.539	-0.523	-0.508	-0.496	-0.483	-0.471	-0.460	-0.448	-0.438	-0.427
35	-0.496	-0.482	-0.468	-0.457	-0.445	-0.434	-0.424	-0.413	-0.404	-0.395
40	-0.466	-0.454	-0.441	-0.430	-0.419	-0.409	-0.399	-0.389	-0.381	-0.372
45	-0.435	-0.423	-0.412	-0.402	-0.392	-0.382	-0.372	-0.362	-0.353	-0.346
50	-0.410	-0.399	-0.389	-0.378	-0.368	-0.359	-0.350	-0.342	-0.334	-0.326

Sample	Significance Level (α)									
Size	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
5	-0.766	-0.730	-0.697	-0.667	-0.635	-0.608	-0.579	-0.554	-0.529	-0.506
10	-0.652	-0.623	-0.595	-0.570	-0.546	-0.522	-0.500	-0.480	-0.459	-0.440
15	-0.559	-0.534	-0.511	-0.489	-0.469	-0.450	-0.431	-0.413	-0.396	-0.379
20	-0.491	-0.470	-0.450	-0.430	-0.412	-0.395	-0.379	-0.363	-0.347	-0.332
25	-0.446	-0.426	-0.407	-0.390	-0.374	-0.358	-0.344	-0.330	-0.316	-0.302
30	-0.408	-0.390	-0.373	-0.357	-0.343	-0.328	-0.314	-0.301	-0.288	-0.276
35	-0.377	-0.361	-0.345	-0.331	-0.317	-0.303	-0.291	-0.279	-0.267	-0.255
40	-0.354	-0.339	-0.325	-0.311	-0.298	-0.285	-0.273	-0.261	-0.250	-0.239
45	-0.331	-0.316	-0.302	-0.290	-0.278	-0.266	-0.255	-0.244	-0.234	-0.224
50	-0.312	-0.298	-0.285	-0.273	-0.261	-0.251	-0.240	-0.228	-0.218	-0.209

Table A.18 Skewness ($\sqrt{b_1}$) Upper Tail Critical Values: $\beta = 3.5$

Sample Size	Significance Level (1-α)									
	0.995	0.99	0.985	0.98	0.975	0.97	0.965	0.96	0.955	0.95
5	1.390	1.335	1.281	1.237	1.197	1.160	1.126	1.094	1.064	1.035
10	1.496	1.337	1.231	1.157	1.100	1.051	1.007	0.970	0.936	0.907
15	1.340	1.177	1.088	1.022	0.964	0.918	0.879	0.845	0.816	0.790
20	1.185	1.048	0.970	0.912	0.868	0.829	0.795	0.764	0.736	0.711
25	1.069	0.949	0.876	0.825	0.781	0.746	0.717	0.691	0.665	0.644
30	0.998	0.887	0.820	0.772	0.734	0.700	0.671	0.643	0.621	0.601
35	0.931	0.827	0.763	0.719	0.683	0.653	0.625	0.603	0.582	0.563
40	0.862	0.766	0.712	0.670	0.640	0.612	0.588	0.566	0.547	0.529
45	0.810	0.724	0.674	0.637	0.608	0.583	0.561	0.540	0.522	0.506
50	0.790	0.700	0.647	0.608	0.576	0.550	0.529	0.510	0.494	0.479

Sample Size	Significance Level (1-α)									
	0.945	0.94	0.935	0.93	0.925	0.92	0.915	0.91	0.905	0.9
5	1.009	0.981	0.956	0.933	0.909	0.886	0.866	0.843	0.823	0.804
10	0.878	0.851	0.827	0.801	0.780	0.759	0.740	0.722	0.705	0.688
15	0.766	0.740	0.719	0.700	0.681	0.663	0.646	0.630	0.614	0.598
20	0.690	0.670	0.651	0.633	0.617	0.602	0.587	0.573	0.558	0.545
25	0.623	0.604	0.587	0.571	0.555	0.541	0.528	0.515	0.503	0.492
30	0.582	0.565	0.550	0.536	0.522	0.509	0.497	0.486	0.475	0.464
35	0.546	0.530	0.516	0.503	0.489	0.477	0.465	0.454	0.444	0.433
40	0.513	0.498	0.486	0.475	0.463	0.451	0.440	0.430	0.420	0.411
45	0.492	0.476	0.463	0.450	0.438	0.428	0.417	0.407	0.398	0.389
50	0.465	0.452	0.439	0.428	0.418	0.407	0.398	0.388	0.380	0.372

Sample Size	Significance Level (1-α)									
	0.89	0.88	0.87	0.86	0.85	0.84	0.83	0.82	0.81	0.8
5	0.765	0.730	0.697	0.663	0.633	0.604	0.575	0.551	0.526	0.502
10	0.656	0.627	0.600	0.574	0.550	0.527	0.505	0.484	0.465	0.444
15	0.571	0.547	0.524	0.500	0.478	0.459	0.439	0.420	0.402	0.386
20	0.520	0.497	0.476	0.455	0.436	0.417	0.400	0.383	0.367	0.352
25	0.470	0.448	0.430	0.411	0.395	0.379	0.363	0.348	0.332	0.319
30	0.444	0.425	0.406	0.388	0.372	0.358	0.343	0.328	0.314	0.301
35	0.415	0.397	0.380	0.364	0.348	0.334	0.321	0.308	0.296	0.284
40	0.393	0.376	0.361	0.345	0.331	0.317	0.304	0.291	0.279	0.268
45	0.373	0.357	0.342	0.327	0.314	0.301	0.289	0.277	0.266	0.256
50	0.355	0.340	0.326	0.314	0.301	0.289	0.278	0.267	0.256	0.246

Table A.19 Skewness ($\sqrt{b_1}$) Lower Tail Critical Values: $\beta = 4.0$

Sample	Significance Level (α)									
Size	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
5	-1.402	-1.346	-1.298	-1.258	-1.221	-1.186	-1.154	-1.123	-1.093	-1.065
10	-1.571	-1.408	-1.298	-1.226	-1.165	-1.118	-1.075	-1.033	-1.002	-0.969
15	-1.376	-1.221	-1.138	-1.078	-1.025	-0.982	-0.947	-0.914	-0.885	-0.858
20	-1.235	-1.104	-1.023	-0.967	-0.922	-0.884	-0.851	-0.822	-0.796	-0.773
25	-1.122	-1.005	-0.932	-0.880	-0.839	-0.806	-0.778	-0.753	-0.732	-0.711
30	-1.018	-0.924	-0.864	-0.818	-0.784	-0.751	-0.725	-0.702	-0.683	-0.665
35	-0.955	-0.864	-0.806	-0.765	-0.732	-0.706	-0.681	-0.659	-0.640	-0.623
40	-0.894	-0.812	-0.760	-0.722	-0.692	-0.670	-0.647	-0.628	-0.610	-0.594
45	-0.850	-0.768	-0.722	-0.687	-0.659	-0.634	-0.613	-0.595	-0.578	-0.562
50	-0.802	-0.734	-0.691	-0.655	-0.629	-0.606	-0.585	-0.567	-0.551	-0.536

Sample	Significance Level (α)									
Size	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
5	-1.040	-1.016	-0.992	-0.968	-0.945	-0.924	-0.903	-0.883	-0.862	-0.843
10	-0.942	-0.917	-0.893	-0.872	-0.850	-0.829	-0.809	-0.789	-0.770	-0.754
15	-0.834	-0.812	-0.790	-0.770	-0.752	-0.734	-0.718	-0.702	-0.687	-0.673
20	-0.752	-0.732	-0.714	-0.696	-0.681	-0.666	-0.651	-0.637	-0.624	-0.611
25	-0.692	-0.675	-0.659	-0.643	-0.629	-0.616	-0.603	-0.591	-0.578	-0.567
30	-0.646	-0.630	-0.614	-0.601	-0.589	-0.576	-0.564	-0.553	-0.541	-0.531
35	-0.607	-0.592	-0.578	-0.566	-0.554	-0.542	-0.531	-0.520	-0.509	-0.501
40	-0.579	-0.564	-0.551	-0.539	-0.528	-0.517	-0.507	-0.497	-0.487	-0.477
45	-0.548	-0.535	-0.522	-0.512	-0.501	-0.490	-0.480	-0.471	-0.462	-0.453
50	-0.524	-0.512	-0.500	-0.489	-0.478	-0.469	-0.460	-0.451	-0.443	-0.435

Sample	Significance Level (α)									
Size	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
5	-0.806	-0.771	-0.737	-0.706	-0.676	-0.647	-0.619	-0.594	-0.569	-0.545
10	-0.723	-0.695	-0.667	-0.640	-0.614	-0.590	-0.567	-0.547	-0.527	-0.507
15	-0.645	-0.620	-0.595	-0.573	-0.553	-0.533	-0.515	-0.496	-0.478	-0.460
20	-0.586	-0.564	-0.542	-0.523	-0.505	-0.487	-0.469	-0.452	-0.437	-0.422
25	-0.545	-0.525	-0.506	-0.488	-0.471	-0.455	-0.440	-0.425	-0.411	-0.396
30	-0.511	-0.493	-0.475	-0.458	-0.442	-0.428	-0.413	-0.399	-0.386	-0.373
35	-0.483	-0.465	-0.449	-0.434	-0.418	-0.405	-0.392	-0.380	-0.368	-0.356
40	-0.460	-0.444	-0.429	-0.415	-0.401	-0.388	-0.376	-0.363	-0.352	-0.341
45	-0.438	-0.423	-0.409	-0.395	-0.382	-0.371	-0.359	-0.348	-0.338	-0.327
50	-0.420	-0.405	-0.392	-0.379	-0.367	-0.356	-0.345	-0.334	-0.323	-0.314

Table A.20 Skewness ($\sqrt{b_1}$) Upper Tail Critical Values: $\beta = 4.0$

Sample Size	Significance Level (1-α)									
	0.995	0.99	0.985	0.98	0.975	0.97	0.965	0.96	0.955	0.95
5	1.382	1.323	1.266	1.218	1.176	1.138	1.101	1.068	1.037	1.008
10	1.430	1.267	1.165	1.092	1.033	0.986	0.943	0.909	0.874	0.843
15	1.243	1.098	1.001	0.934	0.883	0.841	0.799	0.766	0.738	0.713
20	1.083	0.953	0.879	0.823	0.779	0.742	0.708	0.676	0.648	0.626
25	0.973	0.849	0.779	0.726	0.689	0.655	0.624	0.597	0.574	0.553
30	0.891	0.784	0.720	0.673	0.636	0.603	0.573	0.549	0.528	0.507
35	0.820	0.721	0.663	0.619	0.584	0.554	0.527	0.505	0.485	0.466
40	0.752	0.663	0.609	0.567	0.537	0.511	0.487	0.467	0.448	0.431
45	0.698	0.619	0.569	0.534	0.505	0.481	0.458	0.439	0.423	0.405
50	0.678	0.592	0.541	0.504	0.474	0.449	0.429	0.410	0.393	0.378

Sample Size	Significance Level (1-α)									
	0.945	0.94	0.935	0.93	0.925	0.92	0.915	0.91	0.905	0.9
5	0.978	0.952	0.925	0.900	0.877	0.853	0.831	0.808	0.789	0.770
10	0.814	0.787	0.763	0.740	0.719	0.699	0.681	0.661	0.644	0.626
15	0.688	0.665	0.643	0.624	0.606	0.588	0.573	0.554	0.539	0.524
20	0.605	0.585	0.566	0.548	0.533	0.517	0.503	0.489	0.476	0.463
25	0.533	0.515	0.498	0.483	0.468	0.453	0.440	0.427	0.416	0.404
30	0.490	0.472	0.457	0.443	0.430	0.417	0.406	0.394	0.383	0.372
35	0.450	0.435	0.421	0.407	0.394	0.382	0.371	0.361	0.350	0.339
40	0.416	0.403	0.389	0.378	0.366	0.354	0.344	0.334	0.325	0.316
45	0.391	0.377	0.365	0.352	0.340	0.330	0.321	0.312	0.302	0.293
50	0.364	0.351	0.339	0.328	0.318	0.309	0.300	0.291	0.282	0.274

Sample Size	Significance Level (1-α)									
	0.89	0.88	0.87	0.86	0.85	0.84	0.83	0.82	0.81	0.8
5	0.730	0.695	0.661	0.627	0.596	0.567	0.540	0.515	0.492	0.469
10	0.595	0.567	0.538	0.514	0.490	0.467	0.445	0.425	0.405	0.385
15	0.497	0.473	0.449	0.427	0.406	0.386	0.367	0.348	0.331	0.314
20	0.438	0.416	0.394	0.374	0.354	0.337	0.320	0.303	0.287	0.271
25	0.382	0.363	0.343	0.326	0.309	0.293	0.277	0.262	0.248	0.235
30	0.352	0.334	0.317	0.301	0.283	0.268	0.253	0.239	0.226	0.213
35	0.321	0.305	0.288	0.272	0.257	0.242	0.229	0.216	0.204	0.193
40	0.298	0.282	0.266	0.251	0.236	0.222	0.210	0.198	0.186	0.174
45	0.276	0.260	0.246	0.231	0.218	0.205	0.194	0.182	0.171	0.160
50	0.258	0.243	0.229	0.216	0.204	0.192	0.180	0.170	0.159	0.149

Table A.21 Kurtosis (b_2) Lower Tail Critical Values: $\beta = 0.5$

Sample Size	Significance Level (α) 0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
5	1.176	1.185	1.194	1.203	1.211	1.219	1.227	1.235	1.243	1.250
10	1.353	1.443	1.508	1.566	1.623	1.665	1.706	1.741	1.771	1.803
15	1.656	1.802	1.910	2.002	2.082	2.157	2.222	2.283	2.339	2.392
20	2.024	2.210	2.351	2.455	2.553	2.644	2.726	2.803	2.875	2.944
25	2.353	2.577	2.742	2.869	2.990	3.099	3.197	3.286	3.371	3.455
30	2.655	2.908	3.108	3.259	3.402	3.530	3.641	3.747	3.841	3.927
35	2.969	3.294	3.503	3.670	3.820	3.945	4.064	4.179	4.284	4.380
40	3.246	3.561	3.812	3.991	4.140	4.285	4.421	4.542	4.656	4.767
45	3.564	3.886	4.141	4.356	4.524	4.670	4.810	4.933	5.058	5.169
50	3.803	4.166	4.412	4.621	4.806	4.974	5.119	5.256	5.368	5.489

Sample Size	Significance Level (α) 0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
5	1.256	1.264	1.271	1.279	1.287	1.296	1.305	1.314	1.324	1.333
10	1.831	1.865	1.894	1.923	1.954	1.984	2.012	2.044	2.074	2.103
15	2.446	2.497	2.547	2.595	2.642	2.686	2.731	2.775	2.815	2.858
20	3.011	3.074	3.141	3.201	3.255	3.309	3.364	3.416	3.471	3.523
25	3.531	3.608	3.679	3.745	3.810	3.874	3.941	4.002	4.067	4.130
30	4.011	4.097	4.174	4.248	4.320	4.395	4.464	4.535	4.606	4.670
35	4.469	4.561	4.649	4.736	4.820	4.901	4.981	5.058	5.141	5.212
40	4.869	4.965	5.065	5.154	5.236	5.319	5.401	5.483	5.566	5.645
45	5.272	5.373	5.473	5.567	5.662	5.752	5.839	5.931	6.011	6.090
50	5.601	5.710	5.812	5.912	6.016	6.109	6.196	6.289	6.377	6.461

Sample Size	Significance Level (α) 0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
5	1.355	1.377	1.400	1.426	1.451	1.478	1.506	1.534	1.565	1.593
10	2.168	2.229	2.289	2.344	2.400	2.457	2.511	2.568	2.622	2.677
15	2.944	3.024	3.101	3.173	3.246	3.322	3.390	3.461	3.533	3.608
20	3.622	3.721	3.812	3.910	4.006	4.098	4.191	4.279	4.367	4.457
25	4.248	4.368	4.476	4.582	4.688	4.797	4.902	5.007	5.104	5.206
30	4.802	4.932	5.052	5.181	5.298	5.418	5.531	5.645	5.762	5.879
35	5.350	5.491	5.628	5.754	5.885	6.013	6.134	6.258	6.379	6.504
40	5.800	5.944	6.092	6.229	6.367	6.512	6.648	6.774	6.910	7.036
45	6.252	6.403	6.550	6.708	6.848	6.992	7.136	7.271	7.409	7.548
50	6.629	6.787	6.950	7.101	7.265	7.406	7.555	7.707	7.859	8.010

Table A.22 Kurtosis (b_2) Lower Tail Standard Deviations: $\beta = 0.5$

Sample Size	Significance Level (α) 0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
5	0.000	0.001	0.001	0.000	0.001	0.001	0.001	0.001	0.001	0.001
10	0.001	0.003	0.002	0.002	0.003	0.001	0.002	0.001	0.001	0.000
15	0.009	0.011	0.010	0.009	0.008	0.006	0.007	0.008	0.008	0.008
20	0.015	0.006	0.009	0.007	0.004	0.002	0.005	0.009	0.009	0.007
25	0.012	0.008	0.007	0.010	0.008	0.007	0.006	0.007	0.008	0.004
30	0.005	0.005	0.007	0.006	0.009	0.013	0.014	0.011	0.008	0.005
35	0.025	0.038	0.030	0.028	0.029	0.023	0.024	0.024	0.027	0.025
40	0.031	0.017	0.020	0.019	0.016	0.020	0.024	0.025	0.023	0.025
45	0.064	0.043	0.039	0.047	0.051	0.046	0.045	0.044	0.047	0.043
50	0.047	0.045	0.034	0.032	0.036	0.038	0.039	0.039	0.026	0.026

Sample Size	Significance Level (α) 0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
5	0.002	0.001	0.002	0.002	0.003	0.002	0.003	0.003	0.003	0.004
10	0.002	0.000	0.002	0.002	0.002	0.002	0.003	0.003	0.004	0.005
15	0.008	0.006	0.004	0.004	0.004	0.006	0.007	0.009	0.009	0.008
20	0.006	0.005	0.006	0.010	0.009	0.007	0.008	0.008	0.004	0.003
25	0.005	0.004	0.003	0.007	0.008	0.010	0.011	0.010	0.009	0.009
30	0.007	0.003	0.005	0.009	0.007	0.005	0.006	0.005	0.004	0.003
35	0.022	0.023	0.023	0.024	0.024	0.026	0.024	0.025	0.030	0.031
40	0.022	0.020	0.022	0.021	0.016	0.016	0.015	0.016	0.015	0.018
45	0.044	0.045	0.042	0.040	0.040	0.041	0.043	0.042	0.041	0.037
50	0.024	0.021	0.022	0.022	0.022	0.019	0.016	0.014	0.011	0.013

Sample Size	Significance Level (α) 0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
5	0.003	0.003	0.004	0.003	0.004	0.004	0.004	0.005	0.004	0.006
10	0.008	0.008	0.008	0.007	0.007	0.006	0.005	0.007	0.006	0.007
15	0.011	0.009	0.011	0.010	0.009	0.010	0.010	0.009	0.009	0.006
20	0.007	0.009	0.009	0.009	0.008	0.009	0.010	0.010	0.013	0.014
25	0.009	0.010	0.010	0.008	0.009	0.007	0.005	0.007	0.004	0.002
30	0.001	0.006	0.006	0.009	0.008	0.009	0.010	0.008	0.009	0.010
35	0.029	0.030	0.033	0.031	0.031	0.029	0.023	0.023	0.022	0.023
40	0.020	0.018	0.016	0.012	0.012	0.019	0.020	0.015	0.017	0.019
45	0.039	0.036	0.034	0.032	0.023	0.021	0.016	0.015	0.018	0.016
50	0.011	0.010	0.010	0.006	0.010	0.013	0.015	0.018	0.018	0.017

Table A.23 Kurtosis (b_2) Upper Tail Critical Values: $\beta = 0.5$

Sample	Significance Level (1-α)									
Size	0.995	0.99	0.985	0.98	0.975	0.97	0.965	0.96	0.955	0.95
5	3.250	3.249	3.249	3.248	3.247	3.246	3.246	3.244	3.243	3.242
10	8.082	8.059	8.037	8.014	7.992	7.969	7.947	7.921	7.897	7.871
15	12.924	12.822	12.733	12.642	12.563	12.479	12.399	12.316	12.236	12.161
20	17.692	17.463	17.259	17.082	16.913	16.747	16.581	16.427	16.273	16.119
25	22.279	21.881	21.558	21.280	20.998	20.721	20.463	20.223	19.963	19.725
30	26.842	26.261	25.758	25.317	24.883	24.493	24.115	23.775	23.444	23.116
35	31.191	30.334	29.664	29.043	28.512	27.985	27.469	27.004	26.563	26.149
40	35.530	34.507	33.609	32.831	32.119	31.460	30.872	30.281	29.738	29.216
45	39.673	38.253	37.107	36.173	35.363	34.599	33.863	33.174	32.499	31.920
50	43.745	42.046	40.624	39.470	38.486	37.500	36.648	35.870	35.113	34.448

Sample	Significance Level (1-α)									
Size	0.945	0.94	0.935	0.93	0.925	0.92	0.915	0.91	0.905	0.9
5	3.241	3.240	3.238	3.236	3.235	3.233	3.231	3.229	3.227	3.225
10	7.844	7.815	7.789	7.761	7.731	7.699	7.672	7.640	7.609	7.580
15	12.075	11.991	11.906	11.821	11.739	11.661	11.585	11.506	11.425	11.337
20	15.968	15.819	15.670	15.513	15.375	15.237	15.095	14.952	14.822	14.678
25	19.514	19.298	19.060	18.833	18.627	18.414	18.214	17.996	17.805	17.607
30	22.806	22.472	22.199	21.891	21.582	21.310	21.046	20.776	20.519	20.269
35	25.743	25.363	24.975	24.589	24.244	23.878	23.543	23.233	22.924	22.591
40	28.718	28.252	27.810	27.353	26.922	26.540	26.146	25.742	25.377	24.993
45	31.342	30.792	30.241	29.780	29.292	28.829	28.389	27.924	27.523	27.087
50	33.728	33.082	32.457	31.850	31.316	30.772	30.250	29.740	29.236	28.795

Sample	Significance Level (1-α)									
Size	0.89	0.88	0.87	0.86	0.85	0.84	0.83	0.82	0.81	0.8
5	3.220	3.215	3.210	3.204	3.198	3.191	3.184	3.176	3.167	3.159
10	7.519	7.450	7.381	7.313	7.242	7.167	7.095	7.017	6.940	6.866
15	11.176	11.016	10.850	10.686	10.517	10.359	10.198	10.037	9.873	9.718
20	14.403	14.123	13.868	13.615	13.361	13.126	12.886	12.643	12.409	12.179
25	17.220	16.855	16.484	16.145	15.789	15.472	15.140	14.829	14.520	14.226
30	19.782	19.305	18.843	18.403	17.958	17.561	17.168	16.792	16.432	16.097
35	22.007	21.418	20.888	20.333	19.842	19.347	18.895	18.454	18.052	17.657
40	24.264	23.608	22.981	22.391	21.806	21.261	20.709	20.195	19.719	19.267
45	26.286	25.547	24.812	24.128	23.481	22.860	22.273	21.701	21.197	20.707
50	27.911	27.081	26.277	25.558	24.866	24.217	23.588	22.995	22.483	21.984

Table A.24 Kurtosis (b_2) Upper Tail Standard Deviations: $\beta = 0.5$

Sample	Significance Level (1-α)									
Size	0.995	0.99	0.985	0.98	0.975	0.97	0.965	0.96	0.955	0.95
5	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.000
10	0.002	0.002	0.002	0.002	0.002	0.003	0.004	0.003	0.004	0.005
15	0.004	0.009	0.010	0.013	0.012	0.012	0.009	0.012	0.011	0.010
20	0.018	0.011	0.007	0.009	0.013	0.011	0.019	0.016	0.017	0.017
25	0.024	0.036	0.033	0.009	0.018	0.020	0.023	0.021	0.028	0.021
30	0.016	0.018	0.008	0.012	0.014	0.022	0.028	0.035	0.052	0.061
35	0.050	0.072	0.086	0.066	0.059	0.072	0.089	0.099	0.114	0.108
40	0.019	0.039	0.007	0.050	0.059	0.055	0.072	0.059	0.035	0.036
45	0.031	0.100	0.092	0.037	0.019	0.058	0.028	0.062	0.045	0.047
50	0.019	0.054	0.115	0.109	0.072	0.117	0.082	0.084	0.112	0.135

Sample	Significance Level (1-α)									
Size	0.945	0.94	0.935	0.93	0.925	0.92	0.915	0.91	0.905	0.9
5	0.000	0.001	0.000	0.001	0.000	0.001	0.001	0.001	0.001	0.001
10	0.004	0.003	0.004	0.003	0.003	0.002	0.002	0.002	0.003	0.004
15	0.015	0.017	0.020	0.022	0.021	0.020	0.017	0.016	0.015	0.021
20	0.018	0.019	0.021	0.021	0.023	0.014	0.014	0.012	0.011	0.020
25	0.010	0.009	0.020	0.025	0.020	0.022	0.014	0.024	0.023	0.020
30	0.077	0.068	0.060	0.059	0.058	0.061	0.065	0.070	0.075	0.082
35	0.114	0.113	0.105	0.117	0.109	0.113	0.113	0.107	0.090	0.103
40	0.039	0.047	0.044	0.047	0.050	0.060	0.058	0.063	0.061	0.046
45	0.031	0.025	0.061	0.082	0.082	0.098	0.102	0.111	0.118	0.107
50	0.153	0.170	0.156	0.124	0.118	0.108	0.101	0.093	0.091	0.059

Sample	Significance Level (1-α)									
Size	0.89	0.88	0.87	0.86	0.85	0.84	0.83	0.82	0.81	0.8
5	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
10	0.006	0.005	0.003	0.003	0.004	0.003	0.003	0.003	0.004	0.002
15	0.020	0.022	0.025	0.027	0.031	0.030	0.028	0.029	0.033	0.029
20	0.028	0.034	0.035	0.031	0.039	0.031	0.021	0.026	0.027	0.021
25	0.036	0.031	0.026	0.018	0.016	0.013	0.023	0.013	0.014	0.007
30	0.082	0.061	0.065	0.065	0.061	0.056	0.055	0.049	0.060	0.060
35	0.092	0.096	0.077	0.098	0.077	0.091	0.084	0.078	0.072	0.062
40	0.039	0.056	0.067	0.065	0.050	0.058	0.043	0.036	0.025	0.017
45	0.101	0.101	0.078	0.084	0.070	0.063	0.048	0.032	0.034	0.053
50	0.049	0.037	0.056	0.050	0.034	0.033	0.039	0.024	0.004	0.005

Table A.25 Kurtosis (b_2) Lower Tail Critical Values: $\beta = 1.0$

Sample	Significance Level (α)									
Size	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
5	1.185	1.196	1.207	1.216	1.224	1.232	1.239	1.246	1.252	1.257
10	1.296	1.351	1.386	1.419	1.445	1.469	1.491	1.514	1.533	1.551
15	1.447	1.515	1.564	1.600	1.634	1.665	1.692	1.718	1.741	1.764
20	1.590	1.664	1.721	1.766	1.804	1.840	1.872	1.900	1.927	1.954
25	1.712	1.793	1.855	1.904	1.950	1.988	2.027	2.063	2.097	2.131
30	1.808	1.910	1.980	2.039	2.095	2.143	2.184	2.222	2.255	2.290
35	1.921	2.037	2.116	2.181	2.237	2.287	2.333	2.373	2.411	2.448
40	2.014	2.133	2.222	2.286	2.340	2.393	2.441	2.486	2.528	2.567
45	2.114	2.237	2.325	2.402	2.464	2.517	2.568	2.615	2.662	2.705
50	2.204	2.331	2.424	2.499	2.560	2.618	2.669	2.715	2.759	2.798

Sample	Significance Level (α)									
Size	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
5	1.262	1.268	1.274	1.280	1.286	1.292	1.298	1.305	1.311	1.318
10	1.569	1.585	1.600	1.616	1.631	1.644	1.658	1.672	1.685	1.698
15	1.786	1.808	1.828	1.846	1.865	1.883	1.900	1.918	1.935	1.954
20	1.983	2.008	2.033	2.056	2.078	2.099	2.121	2.145	2.166	2.188
25	2.161	2.190	2.219	2.248	2.274	2.300	2.323	2.347	2.373	2.398
30	2.322	2.356	2.384	2.414	2.443	2.471	2.498	2.524	2.549	2.574
35	2.483	2.515	2.548	2.581	2.613	2.645	2.674	2.701	2.729	2.755
40	2.603	2.640	2.674	2.706	2.739	2.772	2.800	2.829	2.857	2.884
45	2.742	2.779	2.813	2.845	2.877	2.908	2.936	2.965	2.993	3.022
50	2.836	2.877	2.912	2.947	2.980	3.013	3.043	3.072	3.101	3.131

Sample	Significance Level (α)									
Size	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
5	1.332	1.346	1.360	1.376	1.391	1.407	1.423	1.439	1.456	1.472
10	1.723	1.747	1.771	1.793	1.816	1.839	1.862	1.884	1.906	1.929
15	1.989	2.019	2.051	2.081	2.114	2.148	2.179	2.213	2.242	2.275
20	2.230	2.270	2.309	2.350	2.387	2.423	2.463	2.498	2.536	2.573
25	2.443	2.484	2.528	2.570	2.613	2.655	2.695	2.735	2.776	2.817
30	2.625	2.678	2.726	2.772	2.816	2.860	2.903	2.949	2.991	3.033
35	2.806	2.859	2.909	2.956	3.003	3.050	3.097	3.142	3.188	3.234
40	2.940	2.992	3.042	3.094	3.149	3.197	3.244	3.289	3.335	3.380
45	3.080	3.134	3.184	3.235	3.285	3.337	3.386	3.434	3.482	3.526
50	3.192	3.248	3.303	3.357	3.410	3.457	3.508	3.557	3.605	3.657

Table A.26 Kurtosis (b_2) Lower Tail Standard Deviation: $\beta = 1.0$

Sample	Significance Level (α)									
Size	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
5	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
10	0.003	0.000	0.002	0.001	0.002	0.002	0.002	0.001	0.002	0.001
15	0.002	0.007	0.004	0.004	0.003	0.003	0.003	0.002	0.003	0.002
20	0.007	0.007	0.005	0.003	0.004	0.003	0.003	0.004	0.004	0.005
25	0.006	0.004	0.004	0.006	0.005	0.006	0.004	0.002	0.004	0.005
30	0.005	0.003	0.002	0.004	0.006	0.008	0.005	0.004	0.003	0.003
35	0.007	0.009	0.008	0.009	0.008	0.008	0.010	0.009	0.008	0.008
40	0.003	0.004	0.008	0.006	0.003	0.003	0.003	0.004	0.004	0.002
45	0.015	0.012	0.013	0.015	0.014	0.013	0.013	0.013	0.015	0.016
50	0.017	0.011	0.012	0.011	0.010	0.010	0.012	0.010	0.009	0.007

Sample	Significance Level (α)									
Size	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
5	0.002	0.002	0.002	0.001	0.001	0.002	0.002	0.002	0.002	0.002
10	0.002	0.002	0.002	0.002	0.001	0.002	0.002	0.002	0.002	0.001
15	0.002	0.002	0.002	0.002	0.002	0.002	0.003	0.002	0.003	0.002
20	0.004	0.004	0.004	0.004	0.005	0.006	0.005	0.004	0.005	0.004
25	0.004	0.003	0.003	0.003	0.003	0.003	0.002	0.002	0.002	0.003
30	0.004	0.005	0.003	0.003	0.002	0.003	0.003	0.003	0.003	0.001
35	0.008	0.007	0.006	0.008	0.008	0.009	0.009	0.009	0.010	0.011
40	0.002	0.003	0.004	0.002	0.003	0.003	0.002	0.002	0.003	0.003
45	0.015	0.015	0.015	0.014	0.013	0.013	0.011	0.009	0.008	0.008
50	0.005	0.007	0.007	0.007	0.007	0.007	0.006	0.005	0.005	0.005

Sample	Significance Level (α)									
Size	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
5	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.002
10	0.001	0.001	0.001	0.002	0.001	0.002	0.002	0.003	0.004	0.004
15	0.002	0.004	0.005	0.006	0.006	0.004	0.003	0.002	0.004	0.003
20	0.003	0.004	0.004	0.003	0.004	0.003	0.004	0.004	0.005	0.004
25	0.002	0.002	0.001	0.001	0.001	0.001	0.002	0.001	0.002	0.002
30	0.003	0.005	0.006	0.005	0.005	0.005	0.004	0.006	0.006	0.007
35	0.011	0.013	0.013	0.013	0.013	0.013	0.012	0.013	0.012	0.014
40	0.002	0.002	0.003	0.003	0.006	0.006	0.007	0.006	0.005	0.003
45	0.009	0.008	0.007	0.008	0.007	0.009	0.008	0.008	0.009	0.005
50	0.005	0.003	0.004	0.003	0.003	0.004	0.003	0.004	0.005	0.005

Table A.27 Kurtosis (b_2) Upper Tail Critical Values: $\beta = 1.0$

Sample Size	Significance Level (1-α)									
	0.995	0.99	0.985	0.98	0.975	0.97	0.965	0.96	0.955	0.95
5	3.231	3.220	3.208	3.196	3.184	3.172	3.160	3.149	3.137	3.125
10	7.473	7.242	7.062	6.917	6.783	6.668	6.556	6.448	6.355	6.250
15	10.876	10.280	9.813	9.470	9.178	8.941	8.708	8.494	8.296	8.114
20	13.601	12.610	11.886	11.395	10.958	10.584	10.251	9.974	9.724	9.463
25	15.524	14.218	13.377	12.751	12.229	11.764	11.383	11.009	10.681	10.353
30	17.572	15.831	14.737	13.879	13.224	12.670	12.168	11.785	11.425	11.094
35	18.674	16.670	15.524	14.657	13.931	13.300	12.800	12.320	11.940	11.574
40	19.964	17.869	16.510	15.500	14.674	14.024	13.487	12.965	12.555	12.180
45	21.008	18.393	16.865	15.827	15.050	14.354	13.806	13.314	12.878	12.517
50	21.684	19.038	17.430	16.294	15.457	14.692	14.063	13.565	13.117	12.679

Sample Size	Significance Level (1-α)									
	0.945	0.94	0.935	0.93	0.925	0.92	0.915	0.91	0.905	0.9
5	3.113	3.102	3.090	3.079	3.067	3.055	3.044	3.032	3.021	3.009
10	6.156	6.063	5.982	5.893	5.813	5.736	5.664	5.595	5.527	5.457
15	7.942	7.785	7.632	7.487	7.362	7.234	7.115	7.003	6.893	6.784
20	9.233	9.028	8.836	8.653	8.475	8.310	8.148	8.011	7.873	7.741
25	10.101	9.836	9.624	9.399	9.190	9.011	8.849	8.689	8.531	8.395
30	10.806	10.533	10.290	10.059	9.856	9.656	9.475	9.292	9.127	8.971
35	11.270	10.984	10.732	10.490	10.273	10.053	9.860	9.690	9.504	9.337
40	11.811	11.482	11.221	10.967	10.738	10.517	10.307	10.098	9.910	9.750
45	12.185	11.859	11.555	11.285	11.060	10.834	10.622	10.423	10.229	10.067
50	12.320	12.007	11.722	11.446	11.197	10.976	10.781	10.571	10.384	10.215

Sample Size	Significance Level (1-α)									
	0.89	0.88	0.87	0.86	0.85	0.84	0.83	0.82	0.81	0.8
5	2.985	2.961	2.936	2.911	2.886	2.861	2.836	2.811	2.785	2.762
10	5.330	5.200	5.076	4.961	4.848	4.742	4.646	4.550	4.458	4.362
15	6.581	6.387	6.205	6.041	5.892	5.754	5.619	5.494	5.379	5.267
20	7.497	7.270	7.061	6.883	6.711	6.557	6.409	6.268	6.136	6.010
25	8.139	7.903	7.680	7.479	7.297	7.125	6.955	6.795	6.652	6.512
30	8.682	8.421	8.183	7.973	7.762	7.578	7.408	7.243	7.084	6.930
35	9.044	8.772	8.538	8.313	8.103	7.897	7.717	7.546	7.383	7.230
40	9.434	9.150	8.891	8.649	8.429	8.213	8.018	7.841	7.676	7.517
45	9.743	9.433	9.168	8.933	8.721	8.504	8.303	8.120	7.947	7.789
50	9.872	9.588	9.328	9.101	8.877	8.661	8.470	8.281	8.105	7.951

Table A.28 Kurtosis (b_2) Upper Tail Standard Deviations: $\beta = 1.0$

Sample Size	Significance Level (1-α)									
	0.995	0.99	0.985	0.98	0.975	0.97	0.965	0.96	0.955	0.95
5	0.001	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002
10	0.007	0.007	0.006	0.003	0.005	0.006	0.008	0.007	0.010	0.005
15	0.044	0.016	0.050	0.053	0.043	0.031	0.035	0.037	0.039	0.042
20	0.027	0.039	0.060	0.022	0.033	0.034	0.035	0.014	0.003	0.016
25	0.143	0.127	0.079	0.035	0.017	0.028	0.013	0.022	0.035	0.052
30	0.088	0.064	0.038	0.041	0.046	0.037	0.050	0.030	0.039	0.023
35	0.169	0.152	0.084	0.022	0.024	0.016	0.038	0.038	0.035	0.043
40	0.113	0.019	0.052	0.086	0.061	0.055	0.070	0.064	0.068	0.060
45	0.110	0.196	0.165	0.088	0.069	0.056	0.041	0.044	0.033	0.037
50	0.096	0.086	0.056	0.036	0.029	0.032	0.037	0.021	0.021	0.048

Sample Size	Significance Level (1-α)									
	0.945	0.94	0.935	0.93	0.925	0.92	0.915	0.91	0.905	0.9
5	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
10	0.009	0.008	0.014	0.016	0.018	0.019	0.015	0.018	0.014	0.016
15	0.039	0.035	0.035	0.041	0.036	0.038	0.035	0.030	0.024	0.022
20	0.009	0.014	0.017	0.011	0.011	0.014	0.013	0.011	0.010	0.005
25	0.043	0.051	0.048	0.050	0.056	0.043	0.035	0.031	0.033	0.025
30	0.010	0.023	0.022	0.029	0.029	0.024	0.030	0.031	0.033	0.041
35	0.030	0.030	0.027	0.026	0.013	0.027	0.021	0.017	0.017	0.020
40	0.053	0.039	0.046	0.056	0.062	0.058	0.050	0.045	0.041	0.039
45	0.036	0.027	0.019	0.022	0.028	0.022	0.029	0.039	0.035	0.037
50	0.042	0.035	0.027	0.035	0.033	0.030	0.019	0.024	0.024	0.019

Sample Size	Significance Level (1-α)									
	0.89	0.88	0.87	0.86	0.85	0.84	0.83	0.82	0.81	0.8
5	0.001	0.001	0.002	0.001	0.001	0.001	0.001	0.001	0.003	0.002
10	0.017	0.018	0.017	0.021	0.018	0.017	0.014	0.010	0.009	0.013
15	0.019	0.022	0.028	0.025	0.020	0.019	0.019	0.016	0.015	0.014
20	0.006	0.010	0.004	0.006	0.008	0.013	0.013	0.013	0.012	0.015
25	0.017	0.014	0.015	0.015	0.011	0.006	0.006	0.010	0.008	0.008
30	0.038	0.029	0.018	0.011	0.008	0.006	0.013	0.016	0.014	0.014
35	0.020	0.024	0.015	0.017	0.017	0.019	0.014	0.013	0.016	0.015
40	0.033	0.039	0.038	0.029	0.028	0.028	0.032	0.032	0.026	0.023
45	0.040	0.035	0.036	0.040	0.039	0.031	0.026	0.022	0.026	0.022
50	0.020	0.015	0.012	0.004	0.007	0.007	0.008	0.005	0.005	0.011

Table A.29 Kurtosis (b_2) Lower Tail Critical Values: $\beta = 1.5$

Sample	Significance Level (α)									
Size	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
5	1.189	1.203	1.214	1.224	1.233	1.240	1.247	1.252	1.258	1.265
10	1.306	1.352	1.386	1.413	1.436	1.457	1.475	1.493	1.510	1.525
15	1.440	1.494	1.532	1.560	1.586	1.607	1.626	1.645	1.662	1.679
20	1.540	1.597	1.637	1.668	1.695	1.718	1.740	1.759	1.775	1.792
25	1.622	1.680	1.719	1.753	1.780	1.804	1.825	1.846	1.864	1.882
30	1.692	1.750	1.794	1.827	1.857	1.883	1.904	1.924	1.946	1.964
35	1.746	1.815	1.862	1.899	1.929	1.954	1.979	2.001	2.021	2.041
40	1.800	1.867	1.912	1.949	1.979	2.008	2.032	2.054	2.076	2.096
45	1.852	1.919	1.965	2.007	2.038	2.066	2.091	2.117	2.139	2.161
50	1.896	1.967	2.014	2.053	2.086	2.114	2.139	2.162	2.184	2.206

Sample	Significance Level (α)									
Size	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
5	1.271	1.277	1.283	1.289	1.295	1.301	1.307	1.313	1.319	1.325
10	1.539	1.552	1.565	1.577	1.590	1.601	1.612	1.623	1.633	1.643
15	1.694	1.708	1.722	1.734	1.746	1.758	1.770	1.781	1.793	1.804
20	1.809	1.823	1.838	1.851	1.865	1.879	1.891	1.903	1.916	1.929
25	1.899	1.916	1.930	1.945	1.960	1.974	1.989	2.001	2.016	2.029
30	1.982	1.999	2.015	2.031	2.046	2.061	2.075	2.089	2.103	2.116
35	2.060	2.077	2.093	2.110	2.126	2.142	2.157	2.173	2.187	2.202
40	2.115	2.136	2.155	2.171	2.188	2.204	2.220	2.235	2.249	2.264
45	2.180	2.199	2.217	2.235	2.252	2.268	2.283	2.297	2.311	2.325
50	2.227	2.246	2.266	2.283	2.298	2.313	2.329	2.346	2.362	2.378

Sample	Significance Level (α)									
Size	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
5	1.338	1.352	1.366	1.379	1.393	1.408	1.422	1.436	1.451	1.466
10	1.663	1.681	1.699	1.716	1.734	1.750	1.767	1.783	1.799	1.816
15	1.826	1.847	1.867	1.886	1.905	1.924	1.944	1.963	1.981	2.000
20	1.953	1.976	1.999	2.021	2.042	2.062	2.083	2.103	2.124	2.143
25	2.054	2.078	2.102	2.126	2.148	2.171	2.192	2.213	2.234	2.256
30	2.143	2.168	2.194	2.219	2.244	2.267	2.289	2.312	2.334	2.356
35	2.229	2.255	2.282	2.307	2.331	2.355	2.380	2.403	2.425	2.448
40	2.292	2.317	2.344	2.368	2.395	2.419	2.443	2.468	2.492	2.514
45	2.353	2.380	2.406	2.432	2.458	2.483	2.509	2.533	2.557	2.581
50	2.408	2.435	2.463	2.489	2.515	2.541	2.566	2.590	2.616	2.641

Table A.30 Kurtosis (b_2) Upper Tail Critical Values: $\beta = 1.5$

Sample Size	Significance Level (1-α)									
	0.995	0.99	0.985	0.98	0.975	0.97	0.965	0.96	0.955	0.95
5	3.197	3.167	3.142	3.116	3.094	3.071	3.052	3.030	3.009	2.988
10	6.523	6.154	5.900	5.690	5.523	5.377	5.249	5.123	5.017	4.909
15	8.416	7.676	7.189	6.869	6.594	6.351	6.141	5.950	5.787	5.648
20	9.620	8.584	7.964	7.511	7.168	6.898	6.647	6.465	6.277	6.122
25	10.061	8.954	8.283	7.859	7.490	7.190	6.934	6.713	6.520	6.347
30	10.696	9.390	8.628	8.110	7.714	7.405	7.137	6.893	6.689	6.507
35	10.763	9.425	8.669	8.171	7.800	7.471	7.202	6.970	6.767	6.593
40	10.894	9.637	8.899	8.335	7.927	7.607	7.338	7.114	6.903	6.713
45	11.004	9.615	8.823	8.322	7.925	7.584	7.306	7.078	6.895	6.722
50	11.024	9.612	8.852	8.313	7.907	7.589	7.301	7.063	6.865	6.699

Sample Size	Significance Level (1-α)									
	0.945	0.94	0.935	0.93	0.925	0.92	0.915	0.91	0.905	0.9
5	2.967	2.950	2.932	2.915	2.898	2.881	2.865	2.848	2.831	2.815
10	4.812	4.718	4.636	4.556	4.489	4.409	4.344	4.278	4.212	4.156
15	5.518	5.392	5.273	5.164	5.076	4.984	4.907	4.831	4.759	4.688
20	5.974	5.828	5.704	5.598	5.493	5.398	5.308	5.219	5.139	5.053
25	6.185	6.043	5.910	5.792	5.683	5.580	5.486	5.401	5.317	5.243
30	6.356	6.222	6.091	5.978	5.864	5.762	5.669	5.575	5.489	5.408
35	6.437	6.292	6.162	6.043	5.931	5.828	5.731	5.638	5.559	5.480
40	6.540	6.383	6.248	6.136	6.023	5.917	5.821	5.730	5.645	5.569
45	6.569	6.428	6.302	6.180	6.070	5.974	5.877	5.790	5.707	5.627
50	6.546	6.407	6.279	6.156	6.046	5.951	5.862	5.781	5.701	5.626

Sample Size	Significance Level (1-α)									
	0.89	0.88	0.87	0.86	0.85	0.84	0.83	0.82	0.81	0.8
5	2.785	2.756	2.728	2.699	2.672	2.644	2.618	2.594	2.569	2.544
10	4.047	3.943	3.848	3.757	3.678	3.599	3.528	3.458	3.395	3.331
15	4.553	4.428	4.319	4.219	4.131	4.044	3.962	3.885	3.815	3.749
20	4.909	4.781	4.662	4.551	4.454	4.357	4.270	4.188	4.115	4.046
25	5.093	4.957	4.837	4.729	4.629	4.534	4.443	4.365	4.288	4.217
30	5.255	5.124	5.001	4.892	4.786	4.689	4.598	4.511	4.435	4.362
35	5.337	5.200	5.079	4.970	4.865	4.767	4.681	4.598	4.519	4.444
40	5.416	5.287	5.168	5.048	4.945	4.851	4.762	4.677	4.601	4.527
45	5.479	5.344	5.224	5.120	5.017	4.921	4.832	4.752	4.675	4.601
50	5.483	5.363	5.250	5.147	5.048	4.956	4.870	4.790	4.709	4.633

Table A.31 Kurtosis (b_2) Lower Tail Critical Values: $\beta = 2.0$

Sample	Significance Level (α)									
Size	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
5	1.192	1.206	1.218	1.228	1.237	1.245	1.251	1.257	1.263	1.269
10	1.313	1.358	1.393	1.421	1.443	1.463	1.482	1.500	1.515	1.529
15	1.449	1.502	1.538	1.565	1.588	1.608	1.625	1.641	1.657	1.671
20	1.539	1.595	1.632	1.659	1.682	1.703	1.721	1.739	1.755	1.770
25	1.615	1.668	1.702	1.732	1.754	1.775	1.794	1.811	1.827	1.842
30	1.676	1.729	1.765	1.792	1.818	1.838	1.856	1.873	1.888	1.903
35	1.723	1.778	1.817	1.847	1.869	1.891	1.909	1.925	1.941	1.956
40	1.765	1.819	1.854	1.885	1.910	1.931	1.950	1.967	1.983	1.998
45	1.801	1.859	1.897	1.925	1.950	1.971	1.989	2.006	2.022	2.037
50	1.839	1.895	1.930	1.959	1.982	2.002	2.019	2.036	2.052	2.067

Sample	Significance Level (α)									
Size	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
5	1.275	1.281	1.287	1.294	1.300	1.306	1.312	1.319	1.325	1.332
10	1.542	1.555	1.566	1.578	1.590	1.600	1.611	1.621	1.630	1.639
15	1.684	1.696	1.709	1.721	1.732	1.742	1.753	1.762	1.772	1.782
20	1.784	1.796	1.808	1.820	1.832	1.843	1.854	1.864	1.873	1.882
25	1.855	1.868	1.881	1.894	1.904	1.915	1.925	1.935	1.945	1.955
30	1.917	1.930	1.943	1.955	1.967	1.977	1.988	1.997	2.007	2.017
35	1.970	1.984	1.997	2.009	2.020	2.032	2.043	2.054	2.064	2.073
40	2.012	2.025	2.039	2.050	2.061	2.071	2.082	2.092	2.102	2.113
45	2.050	2.063	2.075	2.087	2.098	2.109	2.120	2.131	2.141	2.151
50	2.080	2.093	2.106	2.118	2.129	2.140	2.151	2.162	2.172	2.181

Sample	Significance Level (α)									
Size	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
5	1.345	1.358	1.372	1.385	1.399	1.413	1.427	1.441	1.456	1.471
10	1.658	1.676	1.693	1.708	1.724	1.738	1.753	1.768	1.783	1.797
15	1.801	1.819	1.836	1.853	1.870	1.886	1.902	1.917	1.932	1.947
20	1.901	1.919	1.937	1.955	1.971	1.987	2.003	2.019	2.034	2.049
25	1.974	1.993	2.010	2.027	2.044	2.060	2.076	2.092	2.108	2.124
30	2.037	2.055	2.073	2.090	2.108	2.124	2.140	2.156	2.171	2.186
35	2.092	2.111	2.128	2.145	2.163	2.179	2.194	2.210	2.226	2.241
40	2.132	2.150	2.167	2.185	2.201	2.217	2.232	2.248	2.263	2.279
45	2.169	2.186	2.203	2.220	2.237	2.254	2.270	2.286	2.301	2.316
50	2.200	2.220	2.237	2.254	2.271	2.287	2.303	2.318	2.335	2.350

Table A.32 Kurtosis (b_2) Upper Tail Critical Values: $\beta = 2.0$

Sample Size	Significance Level (1-α)									
	0.995	0.99	0.985	0.98	0.975	0.97	0.965	0.96	0.955	0.95
5	3.176	3.133	3.098	3.066	3.037	3.008	2.983	2.959	2.936	2.912
10	5.804	5.394	5.142	4.912	4.747	4.602	4.485	4.379	4.280	4.192
15	6.831	6.139	5.748	5.467	5.219	5.036	4.872	4.734	4.607	4.495
20	7.323	6.455	6.008	5.682	5.438	5.235	5.076	4.935	4.811	4.697
25	7.314	6.529	6.035	5.738	5.478	5.285	5.124	4.977	4.851	4.743
30	7.443	6.619	6.112	5.799	5.550	5.338	5.159	5.012	4.896	4.783
35	7.391	6.528	6.048	5.727	5.489	5.306	5.138	5.006	4.886	4.766
40	7.232	6.500	6.041	5.742	5.499	5.313	5.144	5.014	4.894	4.789
45	7.211	6.411	5.970	5.685	5.446	5.253	5.086	4.959	4.849	4.742
50	7.128	6.342	5.897	5.604	5.388	5.198	5.052	4.914	4.808	4.711

Sample Size	Significance Level (1-α)									
	0.945	0.94	0.935	0.93	0.925	0.92	0.915	0.91	0.905	0.9
5	2.889	2.868	2.847	2.829	2.810	2.792	2.775	2.758	2.741	2.723
10	4.111	4.033	3.967	3.902	3.842	3.782	3.727	3.676	3.628	3.579
15	4.401	4.313	4.236	4.159	4.096	4.032	3.971	3.916	3.864	3.818
20	4.594	4.500	4.421	4.343	4.276	4.207	4.145	4.088	4.033	3.983
25	4.644	4.555	4.470	4.390	4.317	4.251	4.191	4.134	4.081	4.029
30	4.686	4.597	4.518	4.452	4.381	4.314	4.255	4.198	4.144	4.096
35	4.674	4.589	4.512	4.445	4.375	4.311	4.254	4.199	4.147	4.102
40	4.692	4.609	4.535	4.459	4.390	4.325	4.266	4.214	4.166	4.120
45	4.658	4.588	4.517	4.450	4.386	4.327	4.271	4.220	4.173	4.129
50	4.624	4.548	4.480	4.412	4.356	4.302	4.245	4.201	4.153	4.110

Sample Size	Significance Level (1-α)									
	0.89	0.88	0.87	0.86	0.85	0.84	0.83	0.82	0.81	0.8
5	2.692	2.662	2.636	2.610	2.583	2.560	2.537	2.515	2.496	2.477
10	3.498	3.419	3.349	3.284	3.226	3.171	3.118	3.068	3.021	2.979
15	3.726	3.646	3.570	3.495	3.431	3.376	3.323	3.273	3.226	3.180
20	3.882	3.799	3.718	3.647	3.581	3.518	3.462	3.407	3.358	3.311
25	3.937	3.852	3.776	3.708	3.644	3.583	3.526	3.473	3.424	3.379
30	4.001	3.919	3.840	3.771	3.707	3.647	3.591	3.538	3.488	3.444
35	4.015	3.936	3.859	3.790	3.726	3.670	3.618	3.564	3.517	3.473
40	4.031	3.952	3.882	3.815	3.751	3.696	3.643	3.593	3.546	3.503
45	4.045	3.963	3.895	3.831	3.773	3.719	3.666	3.620	3.575	3.529
50	4.030	3.959	3.891	3.831	3.775	3.724	3.675	3.627	3.586	3.541

Table A.33 Kurtosis (b_2) Lower Tail Critical Values: $\beta = 2.5$

Sample	Significance Level (α)									
Size	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
5	1.193	1.207	1.219	1.230	1.239	1.247	1.253	1.260	1.266	1.272
10	1.320	1.368	1.400	1.428	1.451	1.471	1.489	1.506	1.521	1.536
15	1.456	1.512	1.548	1.573	1.597	1.615	1.633	1.650	1.664	1.677
20	1.550	1.602	1.638	1.665	1.688	1.710	1.728	1.744	1.759	1.773
25	1.623	1.673	1.708	1.735	1.757	1.778	1.797	1.813	1.828	1.842
30	1.680	1.731	1.769	1.795	1.820	1.839	1.858	1.874	1.887	1.902
35	1.728	1.782	1.816	1.843	1.865	1.886	1.904	1.921	1.935	1.948
40	1.767	1.820	1.856	1.884	1.907	1.925	1.941	1.957	1.971	1.984
45	1.802	1.858	1.892	1.916	1.939	1.958	1.975	1.991	2.004	2.017
50	1.839	1.886	1.920	1.944	1.966	1.985	2.001	2.016	2.030	2.043

Sample	Significance Level (α)									
Size	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
5	1.278	1.285	1.291	1.296	1.302	1.309	1.316	1.322	1.329	1.335
10	1.549	1.562	1.574	1.584	1.595	1.606	1.618	1.627	1.636	1.645
15	1.691	1.703	1.714	1.726	1.736	1.746	1.756	1.766	1.775	1.785
20	1.786	1.799	1.812	1.822	1.833	1.844	1.854	1.864	1.873	1.882
25	1.855	1.867	1.878	1.889	1.900	1.910	1.921	1.930	1.940	1.949
30	1.914	1.927	1.937	1.948	1.958	1.969	1.979	1.988	1.997	2.006
35	1.961	1.972	1.984	1.995	2.006	2.016	2.026	2.035	2.044	2.053
40	1.997	2.009	2.019	2.030	2.040	2.050	2.060	2.070	2.079	2.088
45	2.029	2.039	2.051	2.061	2.072	2.081	2.091	2.099	2.108	2.116
50	2.056	2.068	2.078	2.088	2.098	2.107	2.116	2.125	2.134	2.142

Sample	Significance Level (α)									
Size	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
5	1.349	1.362	1.376	1.389	1.403	1.417	1.432	1.445	1.460	1.474
10	1.664	1.680	1.696	1.711	1.727	1.742	1.757	1.772	1.786	1.799
15	1.803	1.820	1.837	1.854	1.870	1.886	1.901	1.916	1.930	1.944
20	1.900	1.917	1.933	1.949	1.964	1.979	1.994	2.007	2.021	2.034
25	1.966	1.983	2.000	2.015	2.030	2.045	2.059	2.073	2.086	2.100
30	2.022	2.038	2.054	2.068	2.083	2.098	2.112	2.125	2.138	2.151
35	2.069	2.085	2.100	2.115	2.129	2.144	2.157	2.170	2.183	2.196
40	2.104	2.120	2.134	2.149	2.163	2.176	2.189	2.202	2.215	2.227
45	2.132	2.148	2.163	2.176	2.190	2.203	2.216	2.229	2.242	2.254
50	2.158	2.174	2.189	2.203	2.217	2.229	2.242	2.255	2.267	2.279

Table A.34 Kurtosis (b_2) Upper Tail Critical Values: $\beta = 2.5$

Sample Size	Significance Level (1-α)									
	0.995	0.99	0.985	0.98	0.975	0.97	0.965	0.96	0.955	0.95
5	3.165	3.116	3.079	3.041	3.009	2.982	2.952	2.926	2.904	2.880
10	5.357	4.949	4.713	4.519	4.350	4.232	4.130	4.029	3.951	3.877
15	5.920	5.320	4.973	4.741	4.557	4.400	4.277	4.170	4.069	3.985
20	6.054	5.385	5.049	4.795	4.609	4.463	4.335	4.229	4.132	4.051
25	5.955	5.323	4.977	4.737	4.565	4.417	4.301	4.201	4.111	4.032
30	5.906	5.327	4.969	4.737	4.558	4.420	4.298	4.198	4.105	4.032
35	5.813	5.212	4.875	4.657	4.490	4.355	4.241	4.148	4.062	3.989
40	5.668	5.144	4.825	4.613	4.456	4.330	4.220	4.130	4.053	3.983
45	5.601	5.053	4.746	4.557	4.401	4.272	4.168	4.076	4.008	3.942
50	5.504	4.960	4.671	4.486	4.333	4.217	4.126	4.045	3.972	3.909

Sample Size	Significance Level (1-α)									
	0.945	0.94	0.935	0.93	0.925	0.92	0.915	0.91	0.905	0.9
5	2.858	2.835	2.813	2.793	2.774	2.756	2.737	2.722	2.705	2.688
10	3.807	3.746	3.685	3.636	3.586	3.539	3.494	3.451	3.411	3.374
15	3.910	3.846	3.788	3.733	3.682	3.634	3.589	3.548	3.508	3.471
20	3.978	3.916	3.858	3.801	3.744	3.696	3.653	3.611	3.572	3.533
25	3.958	3.894	3.836	3.783	3.732	3.690	3.648	3.608	3.569	3.532
30	3.959	3.897	3.844	3.793	3.747	3.703	3.659	3.619	3.584	3.550
35	3.928	3.870	3.815	3.765	3.720	3.677	3.636	3.600	3.565	3.532
40	3.918	3.858	3.808	3.758	3.713	3.668	3.630	3.592	3.557	3.526
45	3.879	3.826	3.780	3.734	3.693	3.653	3.616	3.583	3.552	3.522
50	3.853	3.801	3.752	3.710	3.667	3.630	3.596	3.562	3.532	3.503

Sample Size	Significance Level (1-α)									
	0.89	0.88	0.87	0.86	0.85	0.84	0.83	0.82	0.81	0.8
5	2.658	2.630	2.601	2.575	2.553	2.530	2.509	2.492	2.473	2.455
10	3.303	3.238	3.179	3.125	3.075	3.029	2.986	2.944	2.906	2.870
15	3.401	3.339	3.279	3.227	3.179	3.135	3.094	3.057	3.019	2.986
20	3.464	3.403	3.346	3.292	3.245	3.202	3.159	3.121	3.084	3.049
25	3.464	3.403	3.350	3.302	3.257	3.216	3.177	3.141	3.107	3.072
30	3.485	3.427	3.375	3.325	3.278	3.236	3.198	3.162	3.129	3.097
35	3.473	3.417	3.367	3.321	3.278	3.236	3.199	3.166	3.135	3.105
40	3.469	3.416	3.368	3.323	3.282	3.246	3.208	3.174	3.142	3.113
45	3.467	3.418	3.370	3.325	3.286	3.246	3.212	3.180	3.150	3.123
50	3.449	3.400	3.355	3.313	3.278	3.244	3.211	3.179	3.148	3.121

Table A.35 Kurtosis (b_2) Lower Tail Critical Values: $\beta = 3.0$

Sample	Significance Level (α)									
Size	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
5	1.194	1.208	1.220	1.231	1.241	1.248	1.254	1.261	1.267	1.273
10	1.324	1.373	1.406	1.432	1.457	1.477	1.495	1.511	1.526	1.541
15	1.464	1.518	1.553	1.580	1.602	1.623	1.641	1.657	1.671	1.685
20	1.557	1.611	1.647	1.674	1.698	1.718	1.737	1.753	1.768	1.782
25	1.631	1.682	1.717	1.744	1.768	1.787	1.805	1.820	1.837	1.851
30	1.687	1.742	1.780	1.808	1.830	1.850	1.869	1.884	1.899	1.912
35	1.737	1.790	1.826	1.854	1.876	1.896	1.914	1.928	1.943	1.958
40	1.780	1.831	1.868	1.894	1.916	1.935	1.952	1.966	1.981	1.994
45	1.815	1.867	1.899	1.927	1.948	1.968	1.984	1.998	2.013	2.025
50	1.847	1.894	1.930	1.954	1.974	1.992	2.009	2.024	2.038	2.050

Sample	Significance Level (α)									
Size	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
5	1.280	1.286	1.292	1.299	1.305	1.311	1.318	1.325	1.331	1.338
10	1.554	1.567	1.580	1.591	1.601	1.612	1.622	1.632	1.642	1.651
15	1.698	1.711	1.722	1.733	1.744	1.755	1.765	1.775	1.785	1.793
20	1.795	1.807	1.820	1.832	1.843	1.853	1.863	1.872	1.882	1.892
25	1.864	1.876	1.887	1.899	1.909	1.919	1.929	1.939	1.949	1.957
30	1.924	1.935	1.946	1.957	1.967	1.977	1.987	1.996	2.006	2.014
35	1.970	1.982	1.992	2.003	2.013	2.024	2.033	2.042	2.050	2.059
40	2.006	2.018	2.028	2.038	2.048	2.057	2.067	2.076	2.085	2.093
45	2.037	2.047	2.058	2.068	2.078	2.086	2.095	2.104	2.113	2.121
50	2.063	2.073	2.082	2.093	2.102	2.112	2.121	2.129	2.138	2.146

Sample	Significance Level (α)									
Size	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
5	1.351	1.364	1.378	1.392	1.406	1.420	1.434	1.449	1.463	1.477
10	1.669	1.686	1.702	1.718	1.733	1.748	1.763	1.777	1.791	1.805
15	1.813	1.830	1.847	1.864	1.879	1.894	1.909	1.923	1.937	1.951
20	1.910	1.926	1.942	1.957	1.972	1.986	2.000	2.014	2.028	2.042
25	1.974	1.991	2.007	2.022	2.036	2.051	2.066	2.079	2.092	2.106
30	2.031	2.046	2.061	2.076	2.091	2.104	2.118	2.131	2.143	2.156
35	2.075	2.091	2.106	2.121	2.135	2.148	2.161	2.174	2.186	2.198
40	2.109	2.124	2.138	2.153	2.166	2.179	2.192	2.204	2.216	2.228
45	2.137	2.151	2.165	2.179	2.192	2.204	2.217	2.229	2.241	2.252
50	2.161	2.176	2.190	2.204	2.217	2.230	2.242	2.253	2.264	2.275

Table A.36 Kurtosis (b_2) Upper Tail Critical Values: $\beta = 3.0$

Sample	Significance Level (1-α)									
Size	0.995	0.99	0.985	0.98	0.975	0.97	0.965	0.96	0.955	0.95
5	3.161	3.111	3.071	3.034	3.002	2.970	2.944	2.916	2.890	2.867
10	5.171	4.767	4.531	4.366	4.224	4.100	4.000	3.918	3.847	3.775
15	5.444	4.930	4.630	4.421	4.280	4.163	4.052	3.959	3.878	3.804
20	5.413	4.857	4.600	4.407	4.251	4.139	4.032	3.944	3.870	3.802
25	5.222	4.758	4.481	4.294	4.158	4.042	3.948	3.870	3.804	3.746
30	5.131	4.687	4.427	4.248	4.112	4.014	3.925	3.848	3.778	3.723
35	5.026	4.568	4.331	4.158	4.026	3.930	3.848	3.778	3.720	3.666
40	4.879	4.486	4.257	4.096	3.981	3.887	3.812	3.745	3.686	3.635
45	4.805	4.403	4.192	4.039	3.929	3.838	3.753	3.698	3.647	3.598
50	4.692	4.314	4.117	3.980	3.875	3.793	3.721	3.663	3.612	3.569

Sample	Significance Level (1-α)									
Size	0.945	0.94	0.935	0.93	0.925	0.92	0.915	0.91	0.905	0.9
5	2.846	2.824	2.803	2.782	2.763	2.744	2.727	2.710	2.694	2.678
10	3.711	3.652	3.601	3.553	3.510	3.466	3.424	3.385	3.347	3.313
15	3.742	3.688	3.637	3.588	3.547	3.503	3.464	3.433	3.399	3.368
20	3.744	3.692	3.645	3.600	3.559	3.519	3.484	3.451	3.420	3.388
25	3.691	3.641	3.597	3.556	3.522	3.486	3.452	3.421	3.390	3.362
30	3.670	3.622	3.581	3.540	3.505	3.469	3.437	3.408	3.380	3.354
35	3.618	3.579	3.538	3.501	3.464	3.432	3.403	3.373	3.350	3.325
40	3.593	3.552	3.514	3.477	3.443	3.412	3.384	3.359	3.335	3.309
45	3.555	3.515	3.477	3.446	3.417	3.390	3.365	3.340	3.316	3.293
50	3.528	3.493	3.457	3.426	3.395	3.367	3.341	3.315	3.293	3.271

Sample	Significance Level (1-α)									
Size	0.89	0.88	0.87	0.86	0.85	0.84	0.83	0.82	0.81	0.8
5	2.647	2.618	2.592	2.567	2.543	2.522	2.502	2.484	2.466	2.449
10	3.246	3.188	3.135	3.084	3.038	2.993	2.951	2.913	2.877	2.844
15	3.309	3.252	3.202	3.154	3.113	3.074	3.037	3.004	2.971	2.940
20	3.330	3.278	3.230	3.187	3.147	3.112	3.076	3.042	3.008	2.979
25	3.310	3.262	3.219	3.181	3.141	3.109	3.075	3.045	3.016	2.990
30	3.305	3.260	3.218	3.179	3.144	3.111	3.082	3.052	3.023	2.996
35	3.282	3.241	3.202	3.166	3.131	3.101	3.072	3.045	3.020	2.997
40	3.264	3.224	3.188	3.155	3.123	3.093	3.065	3.040	3.015	2.992
45	3.250	3.212	3.178	3.149	3.118	3.090	3.065	3.039	3.015	2.992
50	3.230	3.193	3.160	3.129	3.101	3.075	3.051	3.026	3.004	2.983

Table A.37 Kurtosis (b_2) Lower Tail Critical Values: $\beta = 3.5$

Sample Size	Significance Level (α)									
	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
5	1.194	1.209	1.221	1.232	1.241	1.249	1.255	1.262	1.268	1.274
10	1.329	1.378	1.410	1.437	1.461	1.480	1.498	1.515	1.531	1.545
15	1.469	1.523	1.558	1.584	1.608	1.628	1.645	1.663	1.679	1.692
20	1.563	1.616	1.653	1.683	1.706	1.727	1.745	1.762	1.777	1.791
25	1.639	1.689	1.726	1.754	1.777	1.796	1.814	1.830	1.845	1.860
30	1.693	1.750	1.790	1.818	1.842	1.862	1.880	1.896	1.910	1.924
35	1.745	1.801	1.837	1.865	1.889	1.907	1.925	1.941	1.957	1.970
40	1.790	1.842	1.879	1.906	1.929	1.948	1.964	1.980	1.995	2.008
45	1.824	1.878	1.912	1.939	1.961	1.980	1.998	2.013	2.027	2.039
50	1.853	1.906	1.942	1.967	1.988	2.006	2.023	2.038	2.051	2.064

Sample Size	Significance Level (α)									
	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
5	1.281	1.287	1.294	1.300	1.307	1.313	1.320	1.326	1.333	1.340
10	1.558	1.572	1.584	1.596	1.606	1.617	1.627	1.637	1.646	1.656
15	1.705	1.718	1.729	1.741	1.752	1.763	1.774	1.784	1.793	1.803
20	1.805	1.817	1.830	1.841	1.853	1.863	1.873	1.883	1.892	1.901
25	1.874	1.887	1.899	1.910	1.921	1.931	1.940	1.950	1.960	1.969
30	1.935	1.947	1.960	1.969	1.980	1.990	2.000	2.009	2.018	2.027
35	1.982	1.994	2.005	2.016	2.027	2.037	2.047	2.055	2.064	2.072
40	2.020	2.031	2.042	2.053	2.063	2.073	2.082	2.091	2.100	2.108
45	2.051	2.062	2.072	2.083	2.093	2.102	2.111	2.119	2.128	2.136
50	2.077	2.087	2.098	2.109	2.119	2.128	2.137	2.145	2.154	2.162

Sample Size	Significance Level (α)									
	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
5	1.353	1.366	1.381	1.394	1.408	1.422	1.436	1.451	1.465	1.480
10	1.674	1.691	1.707	1.724	1.739	1.754	1.769	1.784	1.798	1.812
15	1.821	1.840	1.856	1.872	1.888	1.902	1.918	1.933	1.947	1.961
20	1.920	1.936	1.952	1.968	1.983	1.998	2.012	2.026	2.041	2.055
25	1.986	2.001	2.018	2.033	2.048	2.063	2.077	2.091	2.105	2.119
30	2.044	2.060	2.075	2.089	2.103	2.117	2.131	2.144	2.157	2.170
35	2.089	2.106	2.121	2.136	2.150	2.163	2.177	2.189	2.201	2.213
40	2.123	2.138	2.154	2.168	2.182	2.195	2.208	2.220	2.232	2.244
45	2.151	2.166	2.181	2.195	2.208	2.221	2.233	2.245	2.258	2.269
50	2.178	2.192	2.205	2.218	2.232	2.244	2.256	2.268	2.280	2.291

Table A.38 Kurtosis (b_2) Upper Tail Critical Values: $\beta = 3.5$

Sample	Significance Level (1-α)									
Size	0.995	0.99	0.985	0.98	0.975	0.97	0.965	0.96	0.955	0.95
5	3.162	3.110	3.068	3.034	3.000	2.970	2.942	2.915	2.890	2.866
10	5.129	4.737	4.507	4.335	4.202	4.093	4.004	3.915	3.836	3.768
15	5.308	4.833	4.558	4.366	4.222	4.099	4.008	3.928	3.855	3.796
20	5.174	4.738	4.478	4.302	4.167	4.064	3.973	3.897	3.826	3.765
25	5.015	4.578	4.334	4.179	4.052	3.961	3.877	3.810	3.746	3.694
30	4.830	4.450	4.252	4.104	3.994	3.901	3.831	3.764	3.707	3.659
35	4.690	4.328	4.131	4.005	3.901	3.818	3.755	3.694	3.640	3.595
40	4.544	4.239	4.059	3.936	3.837	3.757	3.694	3.640	3.595	3.553
45	4.437	4.140	3.982	3.864	3.778	3.708	3.644	3.596	3.555	3.513
50	4.332	4.068	3.919	3.809	3.724	3.656	3.601	3.554	3.515	3.480

Sample	Significance Level (1-α)									
Size	0.945	0.94	0.935	0.93	0.925	0.92	0.915	0.91	0.905	0.9
5	2.843	2.822	2.801	2.782	2.762	2.743	2.726	2.709	2.693	2.677
10	3.711	3.651	3.597	3.548	3.504	3.461	3.417	3.381	3.346	3.312
15	3.733	3.681	3.631	3.582	3.543	3.503	3.464	3.428	3.395	3.362
20	3.711	3.662	3.613	3.57	3.532	3.496	3.463	3.431	3.402	3.374
25	3.647	3.601	3.562	3.522	3.488	3.457	3.425	3.395	3.37	3.344
30	3.613	3.569	3.533	3.498	3.467	3.437	3.408	3.381	3.357	3.332
35	3.558	3.523	3.488	3.455	3.425	3.395	3.371	3.348	3.324	3.302
40	3.516	3.478	3.447	3.419	3.392	3.366	3.341	3.318	3.295	3.275
45	3.478	3.444	3.414	3.386	3.361	3.337	3.314	3.293	3.273	3.254
50	3.445	3.413	3.387	3.359	3.336	3.313	3.291	3.270	3.251	3.232

Sample	Significance Level (1-α)									
Size	0.89	0.88	0.87	0.86	0.85	0.84	0.83	0.82	0.81	0.8
5	2.647	2.619	2.592	2.567	2.543	2.521	2.502	2.485	2.468	2.450
10	3.248	3.191	3.138	3.089	3.043	3.000	2.958	2.921	2.887	2.854
15	3.306	3.255	3.208	3.165	3.122	3.085	3.047	3.012	2.980	2.949
20	3.322	3.272	3.227	3.187	3.148	3.113	3.079	3.047	3.016	2.988
25	3.298	3.253	3.214	3.176	3.141	3.108	3.077	3.047	3.020	2.994
30	3.285	3.245	3.207	3.169	3.136	3.105	3.076	3.048	3.022	2.998
35	3.260	3.221	3.185	3.152	3.122	3.094	3.067	3.043	3.019	2.996
40	3.235	3.200	3.168	3.138	3.110	3.082	3.057	3.034	3.011	2.990
45	3.218	3.185	3.153	3.124	3.098	3.073	3.049	3.026	3.005	2.985
50	3.198	3.166	3.137	3.108	3.082	3.058	3.036	3.014	2.993	2.974

Table A.39 Kurtosis (b_2) Lower Tail Critical Values: $\beta = 4.0$

Sample	Significance Level (α)									
Size	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
5	1.195	1.209	1.221	1.232	1.241	1.249	1.255	1.262	1.268	1.275
10	1.330	1.380	1.414	1.439	1.464	1.483	1.502	1.518	1.534	1.549
15	1.471	1.528	1.562	1.589	1.612	1.632	1.650	1.667	1.683	1.698
20	1.567	1.620	1.658	1.688	1.713	1.736	1.753	1.769	1.785	1.800
25	1.643	1.697	1.732	1.761	1.785	1.805	1.824	1.839	1.854	1.869
30	1.701	1.759	1.796	1.828	1.852	1.872	1.890	1.907	1.921	1.934
35	1.752	1.811	1.847	1.875	1.898	1.919	1.936	1.952	1.967	1.982
40	1.801	1.853	1.888	1.915	1.940	1.960	1.978	1.993	2.008	2.022
45	1.836	1.890	1.923	1.952	1.974	1.993	2.011	2.026	2.040	2.053
50	1.861	1.917	1.953	1.980	2.002	2.019	2.036	2.052	2.066	2.080

Sample	Significance Level (α)									
Size	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
5	1.282	1.289	1.295	1.301	1.308	1.314	1.321	1.327	1.334	1.340
10	1.562	1.575	1.588	1.600	1.610	1.620	1.631	1.640	1.650	1.660
15	1.710	1.723	1.736	1.748	1.759	1.770	1.780	1.791	1.801	1.811
20	1.814	1.828	1.840	1.851	1.861	1.871	1.881	1.891	1.900	1.910
25	1.884	1.896	1.909	1.920	1.931	1.942	1.951	1.961	1.970	1.979
30	1.948	1.959	1.971	1.981	1.992	2.003	2.013	2.022	2.032	2.041
35	1.996	2.008	2.019	2.029	2.040	2.051	2.061	2.070	2.079	2.088
40	2.034	2.046	2.057	2.068	2.079	2.088	2.097	2.106	2.116	2.125
45	2.065	2.077	2.088	2.099	2.108	2.118	2.127	2.136	2.145	2.154
50	2.092	2.104	2.115	2.126	2.136	2.146	2.155	2.164	2.172	2.180

Sample	Significance Level (α)									
Size	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.2
5	1.354	1.368	1.382	1.396	1.409	1.424	1.438	1.452	1.467	1.482
10	1.677	1.695	1.712	1.728	1.744	1.760	1.775	1.790	1.805	1.818
15	1.829	1.847	1.865	1.880	1.896	1.911	1.926	1.941	1.956	1.971
20	1.928	1.946	1.963	1.979	1.995	2.010	2.025	2.040	2.054	2.069
25	1.997	2.014	2.031	2.046	2.061	2.076	2.091	2.106	2.120	2.133
30	2.058	2.074	2.089	2.104	2.118	2.132	2.146	2.159	2.173	2.187
35	2.105	2.122	2.138	2.153	2.167	2.180	2.194	2.207	2.219	2.231
40	2.141	2.156	2.171	2.186	2.199	2.213	2.226	2.239	2.252	2.264
45	2.170	2.185	2.200	2.214	2.228	2.241	2.253	2.266	2.278	2.289
50	2.196	2.211	2.225	2.238	2.252	2.265	2.278	2.290	2.301	2.313

Table A.40 Kurtosis (b_2) Upper Tail Critical Values: $\beta = 4.0$

Sample Size	Significance Level (1-α)									
	0.995	0.99	0.985	0.98	0.975	0.97	0.965	0.96	0.955	0.95
5	3.162	3.110	3.068	3.036	3.004	2.973	2.943	2.918	2.894	2.870
10	5.191	4.799	4.546	4.374	4.244	4.135	4.027	3.951	3.875	3.807
15	5.383	4.903	4.629	4.430	4.285	4.164	4.069	3.988	3.914	3.848
20	5.233	4.792	4.541	4.363	4.226	4.121	4.034	3.953	3.884	3.819
25	5.050	4.660	4.413	4.251	4.127	4.024	3.938	3.869	3.808	3.755
30	4.849	4.512	4.308	4.156	4.052	3.959	3.887	3.827	3.771	3.720
35	4.677	4.363	4.199	4.066	3.967	3.885	3.813	3.757	3.704	3.658
40	4.547	4.272	4.093	3.981	3.887	3.810	3.746	3.692	3.645	3.602
45	4.423	4.162	4.009	3.911	3.827	3.756	3.697	3.650	3.604	3.564
50	4.358	4.090	3.941	3.840	3.766	3.706	3.652	3.605	3.566	3.531

Sample Size	Significance Level (1-α)									
	0.945	0.94	0.935	0.93	0.925	0.92	0.915	0.91	0.905	0.9
5	2.846	2.825	2.804	2.783	2.765	2.749	2.731	2.713	2.697	2.681
10	3.748	3.686	3.626	3.577	3.528	3.486	3.449	3.408	3.372	3.339
15	3.787	3.731	3.678	3.631	3.588	3.545	3.509	3.472	3.439	3.407
20	3.761	3.711	3.665	3.622	3.583	3.546	3.512	3.477	3.448	3.420
25	3.706	3.661	3.619	3.580	3.543	3.508	3.476	3.446	3.418	3.392
30	3.670	3.629	3.592	3.557	3.523	3.490	3.461	3.432	3.407	3.381
35	3.618	3.579	3.544	3.512	3.481	3.453	3.427	3.401	3.377	3.354
40	3.562	3.527	3.496	3.468	3.441	3.414	3.391	3.370	3.348	3.327
45	3.528	3.495	3.465	3.437	3.411	3.386	3.365	3.343	3.323	3.304
50	3.497	3.466	3.437	3.410	3.386	3.364	3.343	3.322	3.303	3.284

Sample Size	Significance Level (1-α)									
	0.89	0.88	0.87	0.86	0.85	0.84	0.83	0.82	0.81	0.8
5	2.652	2.623	2.596	2.571	2.547	2.525	2.505	2.486	2.471	2.452
10	3.274	3.219	3.164	3.115	3.069	3.027	2.984	2.945	2.910	2.875
15	3.349	3.297	3.247	3.201	3.158	3.120	3.081	3.046	3.011	2.978
20	3.364	3.316	3.271	3.228	3.188	3.150	3.117	3.084	3.054	3.024
25	3.344	3.299	3.259	3.222	3.185	3.152	3.121	3.091	3.062	3.035
30	3.333	3.289	3.252	3.216	3.182	3.150	3.121	3.092	3.066	3.041
35	3.313	3.275	3.238	3.205	3.173	3.142	3.113	3.087	3.062	3.038
40	3.289	3.253	3.219	3.188	3.161	3.133	3.106	3.082	3.058	3.036
45	3.267	3.232	3.201	3.172	3.145	3.119	3.094	3.072	3.049	3.029
50	3.248	3.217	3.186	3.159	3.132	3.107	3.083	3.061	3.040	3.020

Appendix B. Attained Significance Levels

B.1 Tables for Weibull Shape $\beta = 1$

Table B.1 Attained Significance Levels: Sample size = 5 ; $\beta = 1$

	Kurtosis Test Significance Level (α)																			
Skewness Test Significance Level (α)	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20
0.01	0.015	0.024	0.034	0.043	0.053	0.063	0.073	0.083	0.093	0.102	0.112	0.122	0.131	0.141	0.152	0.162	0.171	0.181	0.190	0.200
0.02	0.024	0.029	0.039	0.048	0.058	0.068	0.078	0.088	0.098	0.107	0.117	0.127	0.136	0.146	0.157	0.167	0.176	0.186	0.195	0.205
0.03	0.034	0.039	0.044	0.053	0.062	0.073	0.083	0.093	0.102	0.111	0.121	0.131	0.141	0.151	0.162	0.171	0.181	0.190	0.200	0.209
0.04	0.044	0.048	0.054	0.058	0.067	0.078	0.088	0.098	0.107	0.116	0.126	0.136	0.146	0.156	0.167	0.176	0.186	0.195	0.205	0.214
0.05	0.054	0.059	0.064	0.068	0.073	0.083	0.093	0.103	0.112	0.121	0.131	0.141	0.151	0.161	0.172	0.181	0.191	0.200	0.210	0.219
0.06	0.064	0.069	0.074	0.078	0.083	0.089	0.098	0.108	0.117	0.126	0.136	0.146	0.156	0.166	0.177	0.186	0.196	0.205	0.215	0.224
0.07	0.074	0.079	0.084	0.089	0.093	0.098	0.104	0.112	0.122	0.131	0.141	0.151	0.161	0.171	0.182	0.191	0.201	0.210	0.220	0.229
0.08	0.084	0.088	0.093	0.098	0.102	0.108	0.113	0.119	0.127	0.136	0.146	0.156	0.166	0.176	0.187	0.196	0.206	0.215	0.225	0.234
0.09	0.094	0.099	0.104	0.109	0.113	0.118	0.123	0.129	0.134	0.142	0.152	0.162	0.171	0.181	0.192	0.202	0.211	0.221	0.230	0.240
0.10	0.104	0.109	0.114	0.119	0.123	0.128	0.133	0.139	0.143	0.148	0.157	0.167	0.176	0.186	0.197	0.207	0.216	0.226	0.235	0.245
0.11	0.115	0.119	0.124	0.129	0.133	0.139	0.144	0.149	0.153	0.157	0.164	0.172	0.181	0.191	0.202	0.211	0.221	0.230	0.240	0.249
0.12	0.124	0.129	0.134	0.138	0.143	0.148	0.153	0.158	0.163	0.167	0.172	0.178	0.186	0.195	0.206	0.216	0.225	0.234	0.244	0.253
0.13	0.134	0.138	0.143	0.147	0.152	0.157	0.162	0.167	0.171	0.175	0.180	0.185	0.192	0.199	0.209	0.219	0.228	0.237	0.247	0.256
0.14	0.143	0.147	0.152	0.156	0.160	0.165	0.170	0.175	0.179	0.183	0.188	0.193	0.198	0.205	0.213	0.222	0.231	0.240	0.250	0.259
0.15	0.153	0.157	0.161	0.165	0.170	0.174	0.179	0.184	0.188	0.192	0.197	0.202	0.207	0.212	0.219	0.226	0.234	0.243	0.253	0.262
0.16	0.163	0.166	0.170	0.174	0.178	0.183	0.187	0.192	0.196	0.200	0.205	0.210	0.215	0.220	0.225	0.231	0.239	0.247	0.256	0.265
0.17	0.173	0.177	0.180	0.184	0.187	0.192	0.196	0.201	0.205	0.209	0.214	0.218	0.223	0.228	0.233	0.238	0.244	0.252	0.260	0.268
0.18	0.183	0.186	0.190	0.193	0.197	0.201	0.205	0.210	0.214	0.218	0.222	0.227	0.232	0.236	0.241	0.246	0.251	0.257	0.264	0.272
0.19	0.192	0.195	0.199	0.202	0.206	0.210	0.214	0.219	0.223	0.226	0.231	0.235	0.240	0.245	0.249	0.254	0.258	0.264	0.270	0.277
0.20	0.202	0.205	0.209	0.212	0.215	0.219	0.223	0.228	0.232	0.235	0.240	0.244	0.249	0.253	0.258	0.262	0.266	0.272	0.277	0.283

Table B.2 Attained Significance Levels: Sample size = 10 ; $\beta = 1$

	Kurtosis Test Significance Level (α)																			
Skewness Test Significance Level (α)	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20
0.01	0.015	0.025	0.035	0.044	0.054	0.063	0.072	0.082	0.092	0.102	0.112	0.122	0.131	0.142	0.153	0.163	0.173	0.183	0.193	0.202
0.02	0.025	0.030	0.039	0.049	0.058	0.068	0.077	0.087	0.096	0.107	0.116	0.126	0.135	0.146	0.157	0.167	0.177	0.187	0.196	0.206
0.03	0.035	0.040	0.045	0.053	0.063	0.072	0.081	0.091	0.100	0.111	0.120	0.130	0.139	0.150	0.160	0.170	0.180	0.190	0.200	0.209
0.04	0.044	0.049	0.053	0.058	0.067	0.076	0.085	0.095	0.104	0.114	0.124	0.133	0.142	0.153	0.163	0.173	0.183	0.193	0.202	0.212
0.05	0.054	0.058	0.062	0.067	0.072	0.080	0.089	0.099	0.108	0.118	0.127	0.137	0.146	0.156	0.166	0.176	0.186	0.195	0.205	0.214
0.06	0.063	0.068	0.071	0.076	0.080	0.086	0.093	0.103	0.112	0.122	0.131	0.140	0.149	0.159	0.169	0.179	0.189	0.198	0.208	0.217
0.07	0.073	0.077	0.081	0.085	0.089	0.093	0.098	0.106	0.115	0.125	0.134	0.143	0.152	0.162	0.172	0.181	0.191	0.200	0.210	0.219
0.08	0.083	0.086	0.090	0.094	0.098	0.102	0.105	0.111	0.119	0.128	0.137	0.146	0.155	0.165	0.175	0.184	0.194	0.203	0.212	0.221
0.09	0.091	0.095	0.098	0.102	0.106	0.110	0.113	0.118	0.124	0.131	0.140	0.149	0.157	0.167	0.177	0.186	0.196	0.205	0.214	0.223
0.10	0.101	0.105	0.108	0.111	0.115	0.119	0.122	0.126	0.131	0.137	0.144	0.152	0.160	0.170	0.180	0.189	0.198	0.207	0.216	0.225
0.11	0.111	0.114	0.117	0.120	0.124	0.127	0.131	0.135	0.139	0.143	0.149	0.156	0.163	0.173	0.182	0.191	0.201	0.210	0.219	0.228
0.12	0.120	0.123	0.126	0.130	0.133	0.136	0.139	0.143	0.147	0.151	0.155	0.161	0.167	0.176	0.185	0.194	0.203	0.212	0.221	0.230
0.13	0.129	0.132	0.135	0.138	0.141	0.144	0.147	0.151	0.155	0.159	0.163	0.167	0.172	0.180	0.188	0.197	0.206	0.214	0.223	0.232
0.14	0.140	0.142	0.145	0.148	0.151	0.154	0.157	0.160	0.164	0.168	0.171	0.175	0.179	0.185	0.192	0.200	0.209	0.217	0.226	0.234
0.15	0.149	0.151	0.154	0.157	0.160	0.162	0.165	0.169	0.172	0.175	0.179	0.182	0.186	0.191	0.197	0.204	0.212	0.220	0.228	0.237
0.16	0.159	0.161	0.163	0.166	0.168	0.171	0.174	0.177	0.180	0.184	0.187	0.190	0.193	0.198	0.203	0.208	0.216	0.223	0.231	0.239
0.17	0.169	0.171	0.173	0.176	0.178	0.181	0.183	0.186	0.189	0.192	0.196	0.198	0.202	0.206	0.210	0.214	0.221	0.227	0.235	0.242
0.18	0.179	0.181	0.183	0.185	0.187	0.190	0.192	0.195	0.198	0.201	0.204	0.207	0.210	0.214	0.217	0.221	0.227	0.232	0.239	0.246
0.19	0.188	0.190	0.192	0.194	0.196	0.199	0.201	0.204	0.206	0.209	0.212	0.215	0.218	0.222	0.225	0.229	0.233	0.238	0.244	0.250
0.20	0.198	0.199	0.201	0.203	0.205	0.208	0.210	0.212	0.215	0.218	0.221	0.223	0.226	0.230	0.233	0.236	0.240	0.244	0.249	0.255

Table B.3 Attained Significance Levels: Sample size = 15 ; $\beta = 1$

Skewness Test Significance Level (α)	Kurtosis Test Significance Level (α)																			
	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20
0.01	0.015	0.025	0.036	0.046	0.056	0.066	0.076	0.087	0.097	0.107	0.117	0.127	0.137	0.147	0.156	0.166	0.176	0.185	0.195	0.205
0.02	0.025	0.029	0.040	0.049	0.059	0.069	0.079	0.089	0.099	0.109	0.119	0.129	0.139	0.148	0.158	0.168	0.177	0.187	0.196	0.206
0.03	0.035	0.039	0.044	0.053	0.062	0.072	0.082	0.092	0.101	0.111	0.121	0.131	0.141	0.150	0.160	0.169	0.178	0.188	0.197	0.207
0.04	0.044	0.048	0.052	0.057	0.066	0.075	0.085	0.094	0.104	0.113	0.124	0.133	0.142	0.152	0.161	0.171	0.180	0.189	0.198	0.208
0.05	0.054	0.057	0.061	0.065	0.070	0.078	0.087	0.097	0.106	0.116	0.125	0.135	0.144	0.153	0.163	0.172	0.181	0.191	0.200	0.210
0.06	0.063	0.066	0.070	0.073	0.077	0.082	0.090	0.099	0.109	0.118	0.128	0.137	0.146	0.155	0.165	0.174	0.183	0.192	0.201	0.211
0.07	0.073	0.076	0.079	0.082	0.086	0.090	0.095	0.103	0.111	0.120	0.130	0.139	0.148	0.157	0.166	0.176	0.184	0.194	0.203	0.212
0.08	0.083	0.086	0.089	0.091	0.095	0.098	0.102	0.107	0.115	0.123	0.133	0.141	0.150	0.159	0.168	0.177	0.186	0.195	0.204	0.214
0.09	0.093	0.095	0.098	0.100	0.103	0.106	0.110	0.114	0.119	0.127	0.135	0.143	0.152	0.161	0.170	0.179	0.188	0.197	0.206	0.215
0.10	0.103	0.104	0.107	0.109	0.112	0.115	0.118	0.122	0.125	0.131	0.138	0.146	0.155	0.163	0.172	0.181	0.189	0.198	0.207	0.217
0.11	0.113	0.114	0.116	0.118	0.121	0.124	0.127	0.130	0.133	0.137	0.143	0.150	0.158	0.166	0.174	0.183	0.191	0.200	0.209	0.218
0.12	0.122	0.124	0.126	0.128	0.130	0.132	0.135	0.138	0.141	0.144	0.149	0.154	0.161	0.169	0.177	0.185	0.193	0.202	0.211	0.220
0.13	0.132	0.133	0.135	0.137	0.139	0.141	0.144	0.147	0.149	0.152	0.156	0.160	0.166	0.173	0.180	0.188	0.196	0.204	0.213	0.222
0.14	0.142	0.143	0.144	0.146	0.148	0.150	0.152	0.155	0.157	0.160	0.164	0.167	0.172	0.177	0.184	0.191	0.198	0.206	0.215	0.224
0.15	0.152	0.153	0.154	0.156	0.157	0.159	0.161	0.164	0.166	0.169	0.172	0.175	0.179	0.183	0.189	0.195	0.202	0.209	0.217	0.226
0.16	0.162	0.163	0.164	0.165	0.167	0.169	0.171	0.173	0.175	0.177	0.180	0.183	0.186	0.190	0.195	0.200	0.206	0.213	0.221	0.229
0.17	0.172	0.173	0.174	0.175	0.176	0.178	0.180	0.182	0.184	0.186	0.189	0.191	0.194	0.197	0.202	0.206	0.211	0.217	0.224	0.232
0.18	0.182	0.182	0.183	0.184	0.185	0.187	0.189	0.190	0.192	0.194	0.197	0.199	0.202	0.205	0.209	0.212	0.217	0.222	0.229	0.236
0.19	0.192	0.193	0.193	0.194	0.195	0.197	0.198	0.200	0.202	0.204	0.206	0.208	0.211	0.213	0.216	0.220	0.223	0.228	0.234	0.240
0.20	0.202	0.202	0.203	0.204	0.205	0.206	0.207	0.209	0.210	0.212	0.214	0.217	0.219	0.221	0.224	0.227	0.230	0.234	0.239	0.245

Table B.4 Attained Significance Levels: Sample size = 20 ; $\beta = 1$

Skewness Test Significance Level (α)	Kurtosis Test Significance Level (α)																			
	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20
0.01	0.015	0.024	0.034	0.044	0.054	0.063	0.073	0.082	0.091	0.102	0.112	0.122	0.131	0.141	0.151	0.161	0.172	0.182	0.192	0.202
0.02	0.024	0.029	0.038	0.047	0.056	0.066	0.075	0.084	0.093	0.103	0.114	0.123	0.133	0.142	0.152	0.162	0.172	0.182	0.192	0.202
0.03	0.034	0.037	0.042	0.050	0.059	0.068	0.077	0.085	0.095	0.105	0.115	0.124	0.134	0.143	0.153	0.163	0.173	0.183	0.193	0.203
0.04	0.044	0.047	0.050	0.055	0.062	0.071	0.079	0.088	0.097	0.107	0.117	0.126	0.135	0.144	0.154	0.164	0.174	0.184	0.194	0.203
0.05	0.054	0.056	0.059	0.062	0.066	0.074	0.082	0.090	0.099	0.109	0.118	0.127	0.137	0.146	0.155	0.165	0.175	0.185	0.194	0.204
0.06	0.064	0.066	0.068	0.071	0.074	0.078	0.085	0.093	0.102	0.111	0.120	0.129	0.138	0.147	0.156	0.166	0.176	0.186	0.195	0.205
0.07	0.073	0.075	0.077	0.079	0.082	0.085	0.090	0.096	0.104	0.113	0.122	0.131	0.140	0.149	0.158	0.168	0.177	0.187	0.196	0.206
0.08	0.082	0.084	0.086	0.088	0.090	0.092	0.096	0.101	0.108	0.116	0.125	0.133	0.142	0.151	0.160	0.169	0.179	0.188	0.197	0.207
0.09	0.092	0.093	0.095	0.097	0.099	0.101	0.104	0.107	0.112	0.119	0.128	0.135	0.144	0.153	0.161	0.171	0.180	0.190	0.199	0.208
0.10	0.102	0.103	0.104	0.106	0.107	0.109	0.112	0.115	0.118	0.124	0.131	0.138	0.147	0.155	0.163	0.173	0.182	0.191	0.200	0.209
0.11	0.112	0.112	0.114	0.115	0.116	0.118	0.120	0.123	0.126	0.130	0.136	0.142	0.150	0.157	0.166	0.175	0.184	0.193	0.201	0.211
0.12	0.122	0.123	0.124	0.125	0.126	0.128	0.130	0.132	0.135	0.138	0.142	0.148	0.154	0.161	0.169	0.177	0.186	0.195	0.203	0.212
0.13	0.132	0.132	0.133	0.134	0.135	0.136	0.138	0.140	0.143	0.145	0.149	0.153	0.159	0.165	0.172	0.180	0.188	0.197	0.205	0.214
0.14	0.141	0.142	0.142	0.143	0.144	0.145	0.147	0.149	0.151	0.154	0.157	0.160	0.164	0.169	0.176	0.183	0.191	0.199	0.208	0.216
0.15	0.152	0.152	0.152	0.153	0.154	0.155	0.156	0.158	0.160	0.162	0.165	0.168	0.171	0.175	0.180	0.187	0.195	0.203	0.210	0.219
0.16	0.162	0.162	0.162	0.163	0.163	0.164	0.165	0.167	0.169	0.171	0.173	0.175	0.178	0.182	0.186	0.192	0.199	0.206	0.214	0.222
0.17	0.172	0.172	0.172	0.173	0.173	0.174	0.175	0.177	0.178	0.180	0.182	0.184	0.187	0.189	0.193	0.198	0.204	0.211	0.217	0.225
0.18	0.182	0.182	0.182	0.182	0.183	0.183	0.184	0.185	0.187	0.188	0.190	0.192	0.194	0.197	0.200	0.204	0.209	0.215	0.222	0.228
0.19	0.192	0.192	0.192	0.192	0.193	0.193	0.194	0.195	0.196	0.198	0.200	0.201	0.203	0.205	0.208	0.211	0.216	0.221	0.226	0.233
0.20	0.202	0.202	0.202	0.202	0.202	0.203	0.203	0.204	0.205	0.207	0.208	0.210	0.212	0.213	0.215	0.219	0.222	0.227	0.232	0.238

Table B.5 Attained Significance Levels: Sample size = 25 ; $\beta = 1$

Skewness Test Significance Level (α)	Kurtosis Test Significance Level (α)																			
	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20
0.01	0.015	0.024	0.034	0.043	0.052	0.062	0.071	0.081	0.091	0.101	0.112	0.122	0.132	0.142	0.153	0.163	0.172	0.182	0.192	0.203
0.02	0.024	0.028	0.036	0.045	0.054	0.063	0.072	0.082	0.092	0.102	0.112	0.123	0.132	0.143	0.153	0.164	0.173	0.183	0.193	0.203
0.03	0.033	0.036	0.040	0.047	0.056	0.065	0.074	0.083	0.093	0.103	0.113	0.123	0.133	0.144	0.154	0.164	0.173	0.183	0.193	0.203
0.04	0.042	0.044	0.047	0.051	0.059	0.067	0.076	0.085	0.094	0.104	0.114	0.124	0.134	0.144	0.154	0.165	0.174	0.183	0.193	0.203
0.05	0.052	0.054	0.056	0.058	0.063	0.070	0.078	0.087	0.096	0.106	0.116	0.126	0.135	0.145	0.155	0.165	0.174	0.184	0.194	0.204
0.06	0.062	0.063	0.065	0.067	0.070	0.075	0.081	0.090	0.098	0.108	0.117	0.127	0.136	0.147	0.156	0.166	0.175	0.185	0.195	0.205
0.07	0.072	0.073	0.074	0.076	0.078	0.081	0.086	0.093	0.101	0.110	0.119	0.129	0.138	0.148	0.157	0.167	0.176	0.186	0.195	0.205
0.08	0.082	0.083	0.084	0.085	0.087	0.090	0.093	0.098	0.105	0.113	0.122	0.131	0.139	0.149	0.159	0.169	0.177	0.187	0.196	0.206
0.09	0.091	0.092	0.093	0.094	0.096	0.098	0.100	0.104	0.109	0.116	0.124	0.133	0.141	0.151	0.160	0.170	0.178	0.188	0.197	0.207
0.10	0.102	0.102	0.103	0.104	0.105	0.107	0.109	0.112	0.116	0.121	0.128	0.136	0.144	0.153	0.162	0.172	0.180	0.189	0.198	0.208
0.11	0.112	0.113	0.113	0.114	0.115	0.117	0.119	0.121	0.124	0.128	0.133	0.140	0.148	0.157	0.165	0.174	0.182	0.190	0.200	0.209
0.12	0.122	0.122	0.123	0.123	0.124	0.126	0.127	0.129	0.132	0.135	0.139	0.145	0.152	0.160	0.168	0.176	0.184	0.192	0.201	0.210
0.13	0.132	0.132	0.132	0.133	0.134	0.135	0.136	0.138	0.140	0.143	0.146	0.151	0.156	0.164	0.171	0.179	0.186	0.194	0.203	0.212
0.14	0.142	0.142	0.143	0.143	0.144	0.145	0.146	0.147	0.149	0.151	0.154	0.157	0.162	0.168	0.175	0.182	0.189	0.197	0.205	0.214
0.15	0.152	0.152	0.152	0.153	0.153	0.154	0.155	0.156	0.158	0.160	0.162	0.165	0.168	0.174	0.180	0.186	0.193	0.200	0.208	0.216
0.16	0.161	0.161	0.161	0.162	0.162	0.163	0.164	0.165	0.166	0.168	0.170	0.172	0.175	0.180	0.185	0.191	0.197	0.203	0.211	0.219
0.17	0.171	0.171	0.171	0.171	0.172	0.172	0.173	0.174	0.175	0.177	0.178	0.180	0.183	0.187	0.191	0.196	0.201	0.208	0.215	0.222
0.18	0.180	0.180	0.181	0.181	0.181	0.181	0.182	0.183	0.184	0.185	0.187	0.189	0.191	0.194	0.198	0.202	0.207	0.212	0.219	0.226
0.19	0.191	0.191	0.191	0.191	0.191	0.191	0.192	0.192	0.193	0.194	0.196	0.197	0.199	0.202	0.205	0.209	0.213	0.218	0.224	0.230
0.20	0.201	0.201	0.201	0.201	0.201	0.201	0.202	0.202	0.203	0.204	0.205	0.207	0.208	0.211	0.213	0.216	0.220	0.224	0.229	0.235

Table B.6 Attained Significance Levels: Sample size = 30 ; $\beta = 1$

Skewness Test Significance Level (α)	Kurtosis Test Significance Level (α)																			
	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20
0.01	0.014	0.023	0.032	0.041	0.051	0.062	0.072	0.082	0.091	0.101	0.111	0.121	0.131	0.141	0.150	0.160	0.170	0.180	0.190	0.199
0.02	0.023	0.026	0.034	0.043	0.053	0.063	0.073	0.083	0.092	0.102	0.111	0.122	0.131	0.141	0.150	0.160	0.170	0.180	0.190	0.200
0.03	0.031	0.034	0.038	0.046	0.055	0.064	0.074	0.084	0.093	0.102	0.112	0.122	0.132	0.141	0.151	0.160	0.170	0.180	0.190	0.200
0.04	0.041	0.043	0.045	0.050	0.057	0.066	0.076	0.085	0.094	0.103	0.113	0.123	0.133	0.142	0.151	0.161	0.171	0.180	0.190	0.200
0.05	0.050	0.052	0.054	0.057	0.061	0.069	0.078	0.087	0.096	0.105	0.114	0.124	0.133	0.143	0.152	0.161	0.171	0.181	0.190	0.200
0.06	0.060	0.061	0.063	0.065	0.068	0.074	0.081	0.089	0.098	0.107	0.115	0.125	0.135	0.144	0.153	0.162	0.172	0.182	0.191	0.201
0.07	0.070	0.071	0.072	0.074	0.077	0.080	0.086	0.092	0.100	0.109	0.117	0.127	0.136	0.145	0.154	0.163	0.173	0.182	0.192	0.201
0.08	0.080	0.081	0.082	0.083	0.086	0.088	0.092	0.097	0.104	0.112	0.120	0.129	0.138	0.146	0.155	0.164	0.174	0.183	0.192	0.202
0.09	0.091	0.091	0.092	0.093	0.095	0.097	0.100	0.104	0.109	0.115	0.123	0.132	0.140	0.148	0.157	0.166	0.175	0.184	0.193	0.203
0.10	0.101	0.101	0.101	0.102	0.104	0.106	0.108	0.111	0.115	0.120	0.126	0.135	0.142	0.150	0.159	0.167	0.177	0.186	0.195	0.204
0.11	0.111	0.111	0.111	0.112	0.113	0.115	0.117	0.119	0.122	0.126	0.131	0.138	0.146	0.153	0.161	0.169	0.178	0.187	0.196	0.205
0.12	0.120	0.121	0.121	0.121	0.122	0.124	0.125	0.127	0.130	0.133	0.137	0.143	0.150	0.157	0.164	0.172	0.180	0.189	0.197	0.206
0.13	0.131	0.131	0.131	0.131	0.132	0.133	0.135	0.136	0.138	0.141	0.144	0.149	0.154	0.161	0.168	0.175	0.183	0.191	0.199	0.208
0.14	0.140	0.140	0.141	0.141	0.141	0.142	0.144	0.145	0.147	0.149	0.152	0.156	0.160	0.166	0.172	0.178	0.186	0.194	0.202	0.210
0.15	0.151	0.151	0.151	0.151	0.152	0.152	0.153	0.154	0.156	0.158	0.160	0.163	0.167	0.171	0.177	0.183	0.190	0.197	0.204	0.212
0.16	0.160	0.160	0.160	0.160	0.161	0.161	0.162	0.163	0.165	0.166	0.168	0.171	0.173	0.177	0.182	0.187	0.194	0.200	0.207	0.215
0.17	0.170	0.170	0.170	0.170	0.170	0.171	0.172	0.172	0.174	0.175	0.177	0.179	0.181	0.185	0.188	0.193	0.199	0.205	0.211	0.218
0.18	0.180	0.180	0.180	0.180	0.180	0.180	0.181	0.182	0.183	0.184	0.185	0.187	0.189	0.192	0.195	0.199	0.204	0.209	0.216	0.222
0.19	0.190	0.190	0.190	0.190	0.190	0.190	0.191	0.191	0.192	0.193	0.194	0.196	0.198	0.200	0.203	0.206	0.210	0.215	0.221	0.226
0.20	0.200	0.200	0.200	0.200	0.200	0.200	0.201	0.201	0.202	0.203	0.204	0.205	0.206	0.208	0.211	0.214	0.217	0.221	0.226	0.232

Table B.7 Attained Significance Levels: Sample size = 35 ; $\beta = 1$

		Kurtosis Test Significance Level (α)																			
		0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20
Skewness Test Significance Level (α)	0.01	0.015	0.024	0.033	0.043	0.053	0.063	0.073	0.083	0.093	0.104	0.114	0.123	0.133	0.144	0.154	0.164	0.174	0.184	0.195	0.205
	0.02	0.023	0.027	0.035	0.044	0.054	0.064	0.073	0.084	0.094	0.104	0.114	0.124	0.133	0.144	0.154	0.164	0.174	0.184	0.195	0.205
	0.03	0.032	0.035	0.039	0.047	0.056	0.065	0.075	0.085	0.094	0.105	0.115	0.124	0.134	0.144	0.154	0.165	0.175	0.184	0.195	0.205
	0.04	0.042	0.044	0.046	0.051	0.058	0.067	0.076	0.086	0.095	0.105	0.115	0.125	0.134	0.145	0.155	0.165	0.175	0.185	0.195	0.205
	0.05	0.052	0.054	0.055	0.058	0.063	0.071	0.079	0.088	0.097	0.107	0.117	0.126	0.135	0.145	0.155	0.165	0.175	0.185	0.196	0.206
	0.06	0.062	0.063	0.065	0.067	0.070	0.075	0.082	0.090	0.099	0.108	0.118	0.127	0.136	0.146	0.156	0.166	0.176	0.186	0.196	0.206
	0.07	0.072	0.073	0.074	0.076	0.078	0.081	0.086	0.094	0.102	0.111	0.120	0.129	0.137	0.147	0.157	0.167	0.177	0.186	0.197	0.207
	0.08	0.083	0.083	0.084	0.085	0.087	0.090	0.093	0.099	0.105	0.114	0.122	0.130	0.139	0.149	0.158	0.168	0.178	0.187	0.197	0.207
	0.09	0.093	0.093	0.094	0.095	0.096	0.098	0.101	0.105	0.110	0.117	0.125	0.133	0.141	0.150	0.160	0.169	0.179	0.188	0.198	0.208
	0.10	0.103	0.104	0.104	0.105	0.106	0.108	0.110	0.112	0.117	0.122	0.129	0.136	0.144	0.153	0.161	0.171	0.180	0.189	0.199	0.209
	0.11	0.113	0.113	0.113	0.114	0.115	0.116	0.118	0.120	0.123	0.128	0.134	0.140	0.147	0.155	0.164	0.172	0.182	0.190	0.200	0.210
	0.12	0.123	0.123	0.123	0.123	0.124	0.125	0.127	0.128	0.131	0.134	0.139	0.144	0.151	0.159	0.167	0.175	0.183	0.192	0.201	0.211
	0.13	0.132	0.132	0.132	0.133	0.133	0.134	0.135	0.137	0.139	0.142	0.145	0.150	0.155	0.163	0.170	0.178	0.186	0.194	0.203	0.212
	0.14	0.143	0.143	0.143	0.143	0.143	0.144	0.145	0.146	0.148	0.150	0.153	0.156	0.161	0.168	0.174	0.182	0.189	0.197	0.205	0.214
	0.15	0.153	0.153	0.153	0.153	0.154	0.154	0.155	0.156	0.157	0.159	0.161	0.164	0.168	0.174	0.179	0.186	0.193	0.200	0.208	0.216
	0.16	0.163	0.163	0.163	0.163	0.164	0.164	0.165	0.166	0.167	0.168	0.170	0.172	0.176	0.180	0.185	0.191	0.197	0.204	0.211	0.219
	0.17	0.173	0.173	0.173	0.173	0.173	0.174	0.174	0.175	0.176	0.177	0.178	0.180	0.183	0.187	0.191	0.196	0.202	0.208	0.215	0.222
	0.18	0.183	0.183	0.183	0.183	0.183	0.184	0.184	0.184	0.185	0.186	0.188	0.189	0.192	0.195	0.198	0.202	0.207	0.213	0.219	0.226
	0.19	0.194	0.194	0.194	0.194	0.194	0.194	0.194	0.195	0.195	0.196	0.197	0.198	0.200	0.203	0.206	0.210	0.214	0.219	0.225	0.231
	0.20	0.204	0.204	0.204	0.204	0.204	0.204	0.204	0.204	0.205	0.205	0.206	0.208	0.209	0.211	0.214	0.217	0.221	0.225	0.230	0.236

Table B.8 Attained Significance Levels: Sample size = 40 ; $\beta = 1$

		Kurtosis Test Significance Level (α)																			
		0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20
Skewness Test Significance Level (α)	0.01	0.013	0.021	0.031	0.041	0.050	0.059	0.069	0.080	0.090	0.100	0.110	0.120	0.130	0.140	0.150	0.160	0.170	0.181	0.190	0.200
	0.02	0.021	0.025	0.033	0.042	0.051	0.060	0.070	0.080	0.090	0.100	0.110	0.120	0.130	0.140	0.150	0.160	0.170	0.181	0.190	0.200
	0.03	0.030	0.032	0.037	0.044	0.052	0.061	0.070	0.081	0.090	0.101	0.111	0.121	0.130	0.141	0.151	0.161	0.170	0.181	0.191	0.200
	0.04	0.040	0.041	0.044	0.048	0.055	0.063	0.072	0.082	0.091	0.102	0.112	0.121	0.131	0.141	0.151	0.161	0.171	0.181	0.191	0.200
	0.05	0.049	0.050	0.052	0.055	0.059	0.066	0.074	0.083	0.093	0.103	0.112	0.122	0.132	0.141	0.151	0.161	0.171	0.181	0.191	0.200
	0.06	0.058	0.059	0.060	0.062	0.065	0.070	0.077	0.086	0.094	0.104	0.113	0.123	0.132	0.142	0.152	0.162	0.171	0.182	0.191	0.201
	0.07	0.068	0.068	0.069	0.071	0.073	0.076	0.081	0.089	0.097	0.106	0.115	0.124	0.133	0.143	0.152	0.162	0.172	0.182	0.192	0.201
	0.08	0.078	0.078	0.079	0.080	0.081	0.084	0.087	0.093	0.100	0.108	0.117	0.126	0.134	0.144	0.153	0.163	0.172	0.183	0.192	0.201
	0.09	0.088	0.088	0.088	0.089	0.090	0.092	0.095	0.099	0.104	0.111	0.119	0.128	0.136	0.145	0.155	0.164	0.173	0.184	0.193	0.202
	0.10	0.098	0.098	0.098	0.099	0.100	0.101	0.103	0.106	0.111	0.116	0.123	0.131	0.139	0.148	0.156	0.166	0.175	0.185	0.194	0.203
	0.11	0.108	0.108	0.108	0.109	0.109	0.110	0.112	0.115	0.118	0.122	0.128	0.135	0.142	0.150	0.158	0.167	0.176	0.186	0.195	0.204
	0.12	0.118	0.118	0.118	0.119	0.119	0.120	0.122	0.123	0.126	0.129	0.134	0.140	0.147	0.154	0.161	0.170	0.178	0.187	0.196	0.205
	0.13	0.129	0.129	0.129	0.129	0.129	0.130	0.131	0.133	0.135	0.138	0.141	0.146	0.152	0.158	0.165	0.173	0.180	0.189	0.198	0.206
	0.14	0.139	0.139	0.139	0.139	0.139	0.140	0.141	0.142	0.144	0.146	0.149	0.152	0.157	0.163	0.169	0.176	0.183	0.192	0.200	0.208
	0.15	0.150	0.150	0.150	0.150	0.150	0.151	0.151	0.152	0.154	0.155	0.158	0.161	0.164	0.169	0.175	0.181	0.187	0.195	0.203	0.211
	0.16	0.160	0.160	0.160	0.160	0.160	0.160	0.161	0.162	0.163	0.164	0.166	0.168	0.171	0.175	0.180	0.186	0.192	0.199	0.206	0.213
	0.17	0.169	0.169	0.169	0.169	0.170	0.170	0.170	0.171	0.172	0.173	0.175	0.176	0.179	0.182	0.186	0.191	0.197	0.203	0.210	0.216
	0.18	0.179	0.179	0.179	0.179	0.179	0.179	0.179	0.180	0.181	0.182	0.183	0.184	0.187	0.189	0.193	0.197	0.202	0.208	0.214	0.220
	0.19	0.188	0.188	0.188	0.188	0.189	0.189	0.189	0.189	0.190	0.191	0.192	0.193	0.195	0.197	0.200	0.204	0.208	0.213	0.218	0.224
	0.20	0.199	0.199	0.199	0.199	0.199	0.199	0.199	0.200	0.200	0.201	0.202	0.203	0.204	0.206	0.208	0.211	0.215	0.220	0.224	0.230

Table B.9 Attained Significance Levels: Sample size = 45 ; $\beta = 1$

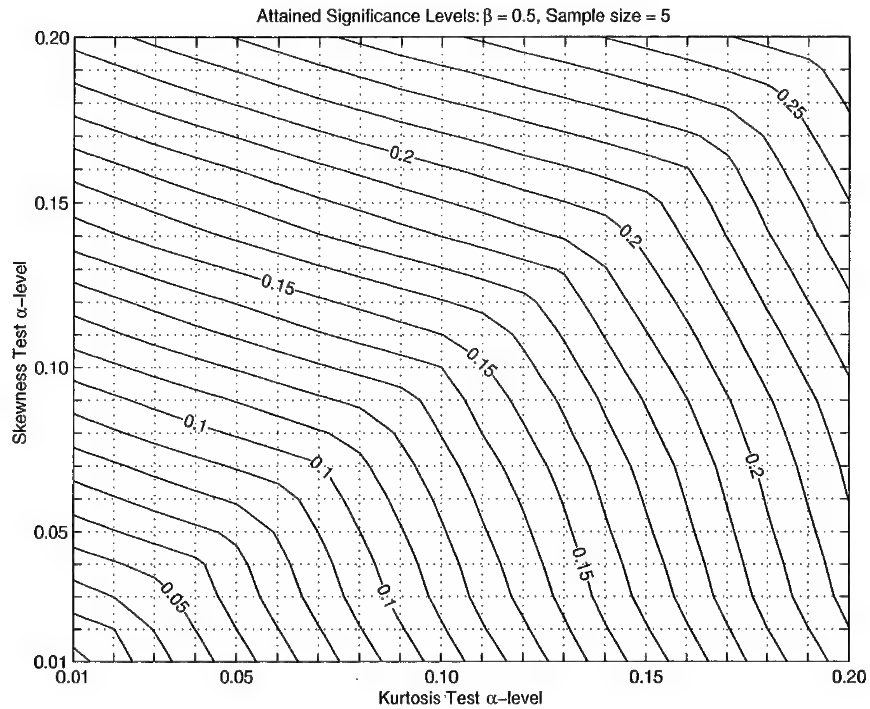
Skewness Test Significance Level (α)	Kurtosis Test Significance Level (α): 0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20
0.01	0.014	0.023	0.033	0.043	0.053	0.063	0.072	0.083	0.093	0.104	0.113	0.123	0.134	0.143	0.153	0.163	0.173	0.183	0.192	0.201
0.02	0.023	0.027	0.035	0.044	0.054	0.063	0.073	0.083	0.094	0.104	0.113	0.123	0.134	0.143	0.153	0.163	0.173	0.183	0.192	0.201
0.03	0.033	0.035	0.039	0.047	0.056	0.065	0.074	0.084	0.095	0.105	0.114	0.124	0.134	0.143	0.153	0.163	0.173	0.183	0.192	0.201
0.04	0.043	0.044	0.047	0.052	0.059	0.067	0.076	0.086	0.096	0.106	0.115	0.125	0.135	0.144	0.154	0.164	0.174	0.183	0.193	0.201
0.05	0.053	0.054	0.055	0.058	0.063	0.070	0.078	0.087	0.097	0.107	0.115	0.125	0.135	0.144	0.154	0.164	0.174	0.183	0.193	0.202
0.06	0.063	0.063	0.065	0.067	0.070	0.075	0.082	0.090	0.099	0.108	0.117	0.126	0.136	0.145	0.155	0.164	0.174	0.184	0.193	0.202
0.07	0.074	0.074	0.075	0.076	0.079	0.082	0.087	0.094	0.102	0.110	0.118	0.128	0.137	0.146	0.156	0.165	0.175	0.184	0.194	0.202
0.08	0.083	0.084	0.084	0.085	0.087	0.090	0.093	0.099	0.106	0.113	0.121	0.129	0.139	0.148	0.157	0.166	0.176	0.185	0.194	0.203
0.09	0.094	0.094	0.094	0.095	0.096	0.098	0.101	0.105	0.110	0.117	0.124	0.132	0.141	0.149	0.158	0.167	0.177	0.186	0.195	0.204
0.10	0.103	0.103	0.103	0.104	0.105	0.106	0.108	0.111	0.116	0.121	0.127	0.135	0.143	0.151	0.160	0.169	0.178	0.187	0.196	0.204
0.11	0.112	0.112	0.112	0.113	0.114	0.115	0.116	0.119	0.122	0.127	0.132	0.139	0.146	0.154	0.162	0.170	0.180	0.188	0.197	0.205
0.12	0.123	0.123	0.123	0.123	0.124	0.125	0.126	0.128	0.130	0.134	0.138	0.144	0.151	0.157	0.165	0.173	0.181	0.190	0.198	0.206
0.13	0.132	0.132	0.132	0.133	0.133	0.134	0.135	0.136	0.138	0.141	0.145	0.150	0.155	0.161	0.168	0.176	0.184	0.192	0.200	0.208
0.14	0.142	0.142	0.142	0.142	0.142	0.143	0.144	0.145	0.147	0.149	0.152	0.156	0.161	0.166	0.172	0.179	0.187	0.194	0.202	0.209
0.15	0.153	0.153	0.153	0.153	0.153	0.153	0.154	0.155	0.156	0.158	0.161	0.164	0.168	0.172	0.178	0.184	0.191	0.197	0.205	0.212
0.16	0.162	0.162	0.162	0.162	0.162	0.163	0.163	0.164	0.165	0.166	0.168	0.171	0.174	0.178	0.183	0.188	0.195	0.201	0.208	0.214
0.17	0.172	0.172	0.172	0.172	0.172	0.172	0.172	0.173	0.174	0.175	0.177	0.179	0.181	0.185	0.189	0.194	0.200	0.205	0.212	0.218
0.18	0.182	0.182	0.182	0.182	0.182	0.182	0.182	0.183	0.184	0.185	0.186	0.187	0.190	0.192	0.196	0.200	0.205	0.210	0.216	0.222
0.19	0.191	0.191	0.191	0.191	0.192	0.192	0.192	0.192	0.193	0.194	0.195	0.196	0.198	0.200	0.203	0.207	0.211	0.216	0.221	0.226
0.20	0.202	0.202	0.202	0.202	0.202	0.202	0.202	0.202	0.203	0.203	0.204	0.205	0.207	0.208	0.211	0.214	0.218	0.222	0.226	0.231

Table B.10 Attained Significance Levels: Sample size = 50 ; $\beta = 1$

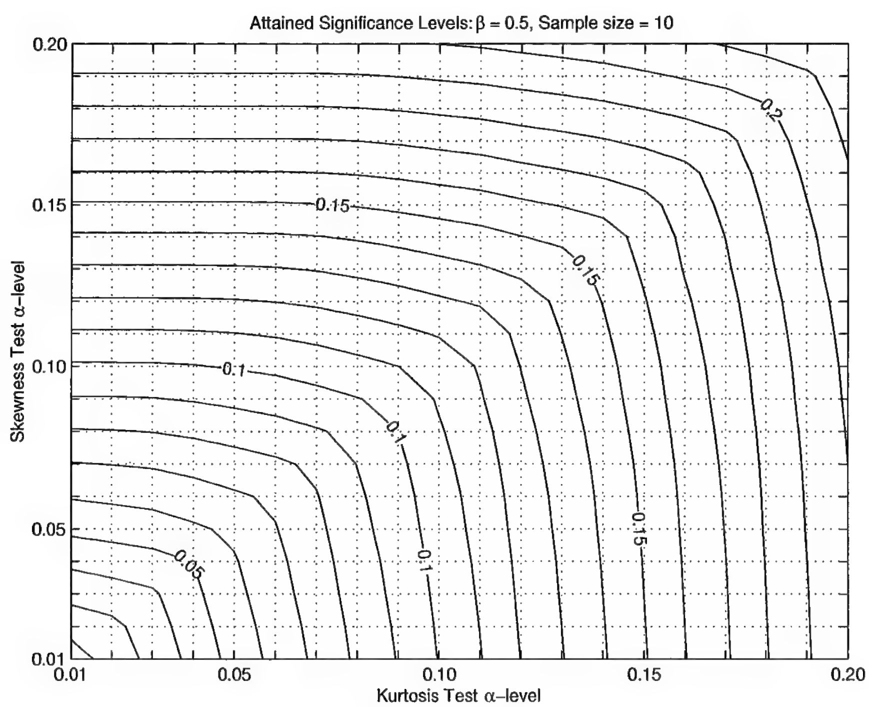
Skewness Test Significance Level (α)	Kurtosis Test Significance Level (α): 0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20
0.01	0.014	0.023	0.032	0.043	0.053	0.063	0.074	0.083	0.093	0.103	0.113	0.124	0.134	0.144	0.154	0.164	0.173	0.182	0.192	0.201
0.02	0.022	0.026	0.034	0.044	0.054	0.064	0.074	0.084	0.094	0.103	0.113	0.124	0.134	0.144	0.154	0.164	0.173	0.182	0.192	0.201
0.03	0.033	0.034	0.038	0.047	0.055	0.065	0.076	0.085	0.094	0.104	0.114	0.124	0.134	0.144	0.154	0.164	0.173	0.183	0.192	0.201
0.04	0.042	0.043	0.046	0.051	0.058	0.067	0.077	0.086	0.095	0.105	0.114	0.125	0.134	0.145	0.154	0.165	0.173	0.183	0.192	0.201
0.05	0.052	0.053	0.055	0.058	0.063	0.070	0.079	0.087	0.097	0.106	0.115	0.126	0.135	0.145	0.155	0.165	0.173	0.183	0.192	0.202
0.06	0.063	0.064	0.065	0.067	0.070	0.075	0.083	0.090	0.099	0.107	0.117	0.127	0.136	0.146	0.155	0.165	0.174	0.183	0.192	0.202
0.07	0.073	0.073	0.074	0.075	0.077	0.081	0.087	0.093	0.101	0.109	0.118	0.128	0.137	0.147	0.156	0.166	0.174	0.184	0.193	0.202
0.08	0.083	0.083	0.084	0.085	0.086	0.089	0.093	0.098	0.105	0.112	0.121	0.130	0.138	0.148	0.157	0.167	0.175	0.184	0.193	0.203
0.09	0.093	0.093	0.094	0.095	0.096	0.097	0.100	0.104	0.109	0.116	0.124	0.132	0.140	0.149	0.158	0.168	0.176	0.185	0.194	0.203
0.10	0.103	0.103	0.103	0.104	0.105	0.106	0.108	0.111	0.115	0.121	0.127	0.135	0.143	0.151	0.160	0.169	0.177	0.186	0.195	0.204
0.11	0.113	0.113	0.113	0.114	0.114	0.115	0.117	0.119	0.122	0.126	0.132	0.139	0.146	0.154	0.162	0.171	0.178	0.187	0.196	0.205
0.12	0.124	0.124	0.124	0.124	0.124	0.125	0.126	0.128	0.130	0.133	0.138	0.144	0.150	0.158	0.165	0.173	0.180	0.189	0.197	0.206
0.13	0.134	0.134	0.134	0.134	0.134	0.135	0.136	0.137	0.139	0.141	0.145	0.150	0.156	0.162	0.168	0.176	0.183	0.191	0.199	0.208
0.14	0.144	0.144	0.144	0.144	0.144	0.144	0.145	0.146	0.148	0.149	0.152	0.157	0.161	0.167	0.172	0.180	0.185	0.193	0.201	0.209
0.15	0.154	0.154	0.154	0.154	0.154	0.154	0.155	0.156	0.157	0.159	0.161	0.164	0.168	0.172	0.177	0.184	0.190	0.197	0.204	0.212
0.16	0.164	0.164	0.164	0.164	0.164	0.165	0.165	0.166	0.167	0.168	0.170	0.172	0.176	0.179	0.183	0.190	0.194	0.201	0.207	0.214
0.17	0.175	0.175	0.175	0.175	0.175	0.175	0.175	0.176	0.177	0.178	0.179	0.181	0.184	0.187	0.190	0.196	0.200	0.206	0.212	0.218
0.18	0.184	0.184	0.184	0.184	0.184	0.185	0.185	0.185	0.186	0.187	0.188	0.190	0.192	0.194	0.197	0.202	0.205	0.211	0.216	0.222
0.19	0.194	0.194	0.194	0.194	0.194	0.194	0.194	0.194	0.195	0.196	0.196	0.198	0.200	0.202	0.204	0.208	0.212	0.216	0.221	0.226
0.20	0.204	0.204	0.204	0.204	0.204	0.204	0.204	0.204	0.205	0.205	0.206	0.207	0.208	0.210	0.212	0.216	0.219	0.223	0.227	0.232

Appendix C. Contour Plots for Attained Significance Levels

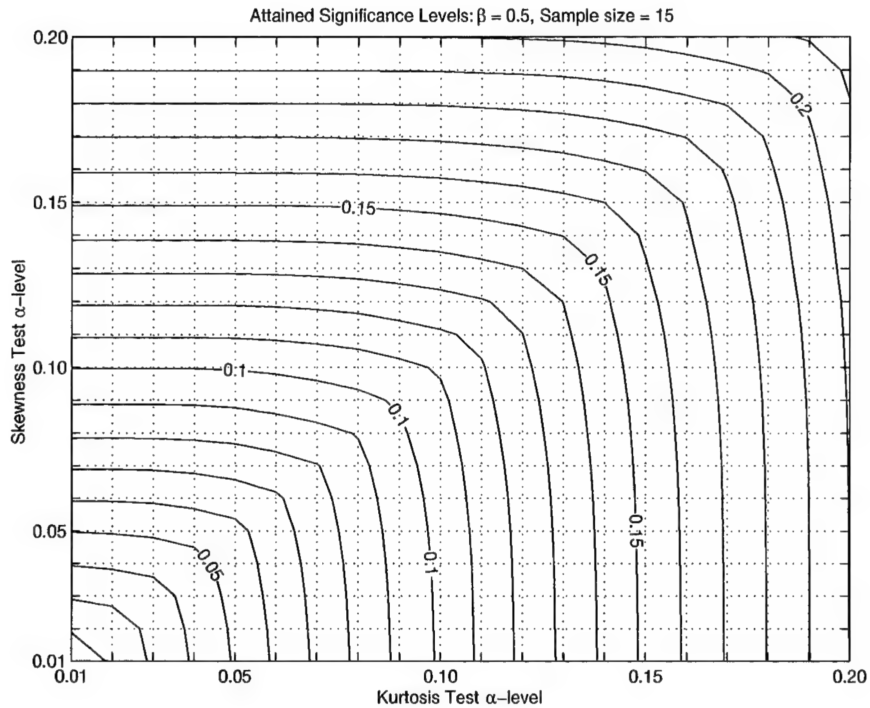
C.1 Weibull Shape $\beta = 0.5$



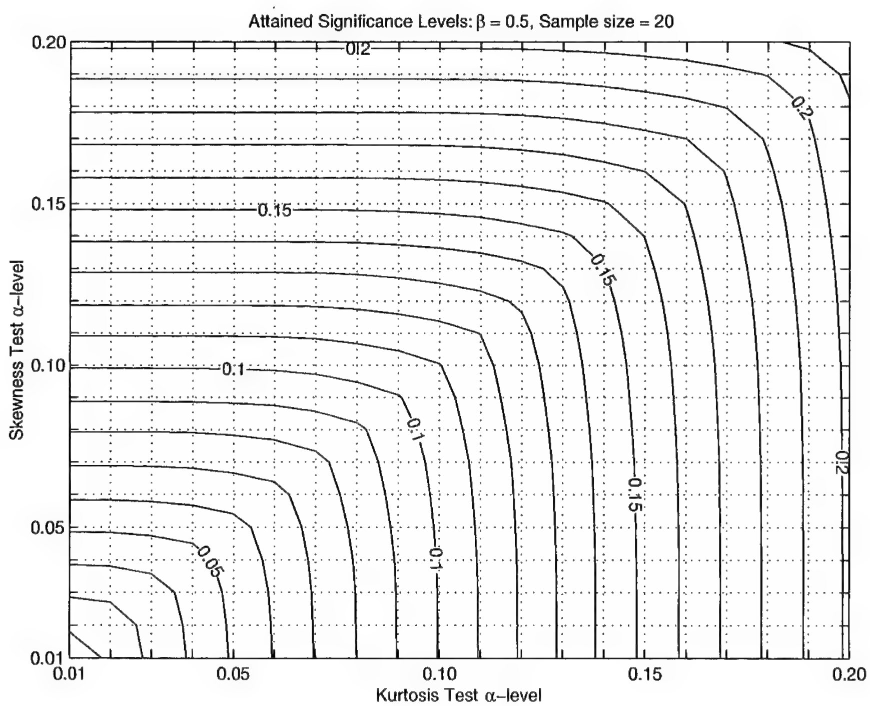
(a) Sample Size 5



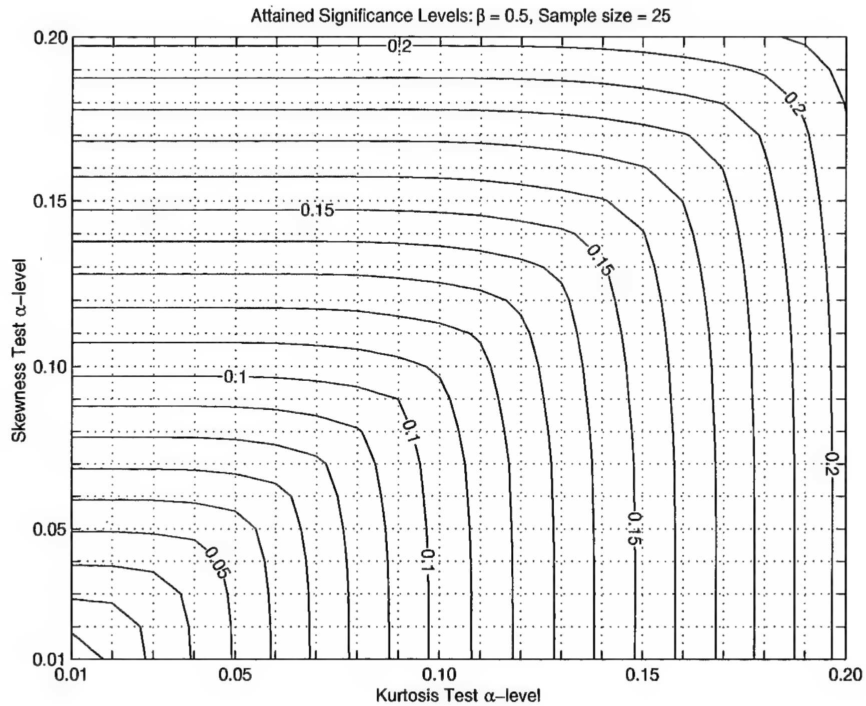
(b) Sample Size 10



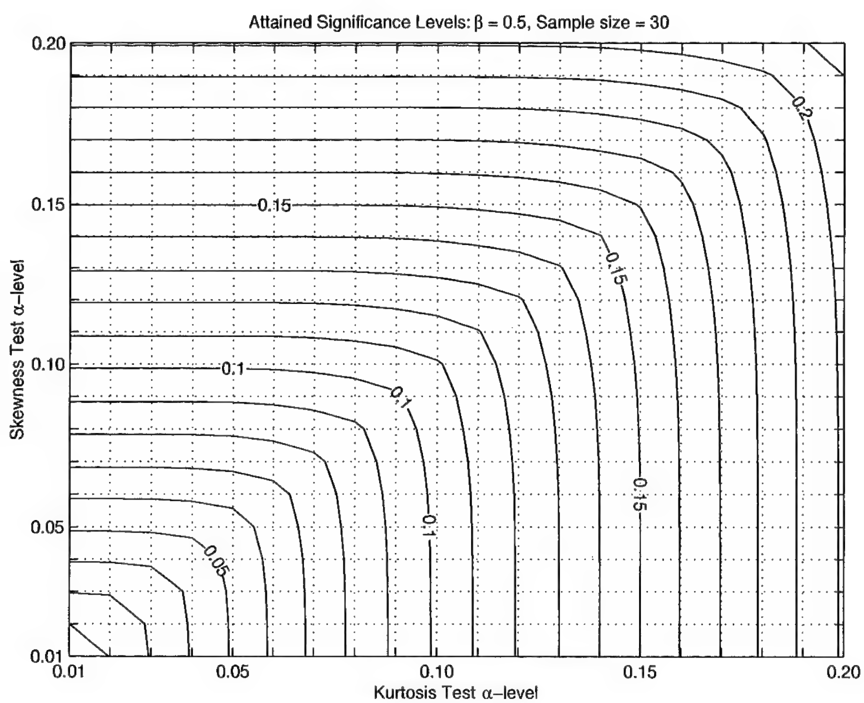
(c) Sample Size 15



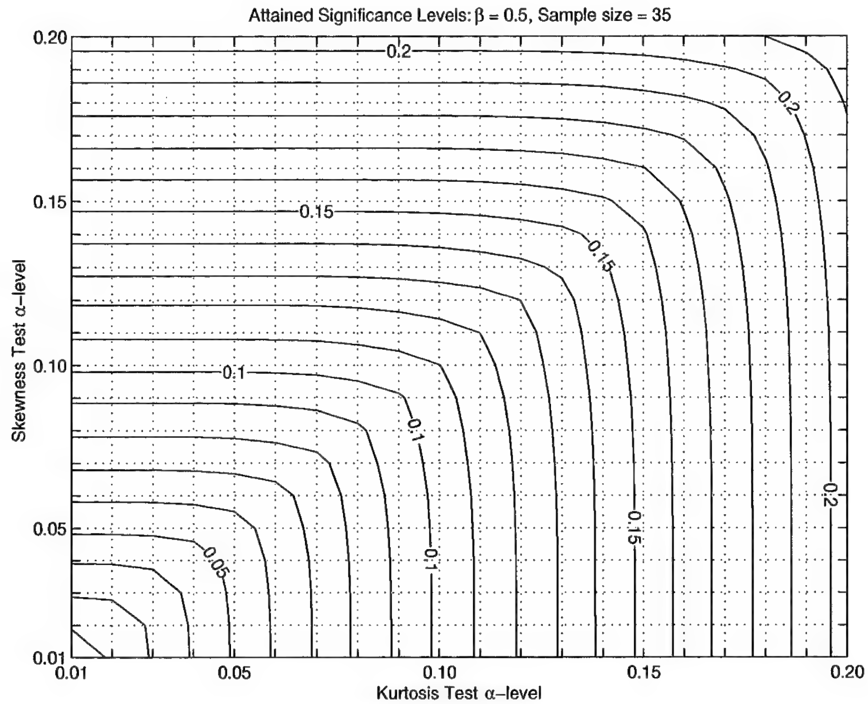
(d) Sample Size 20



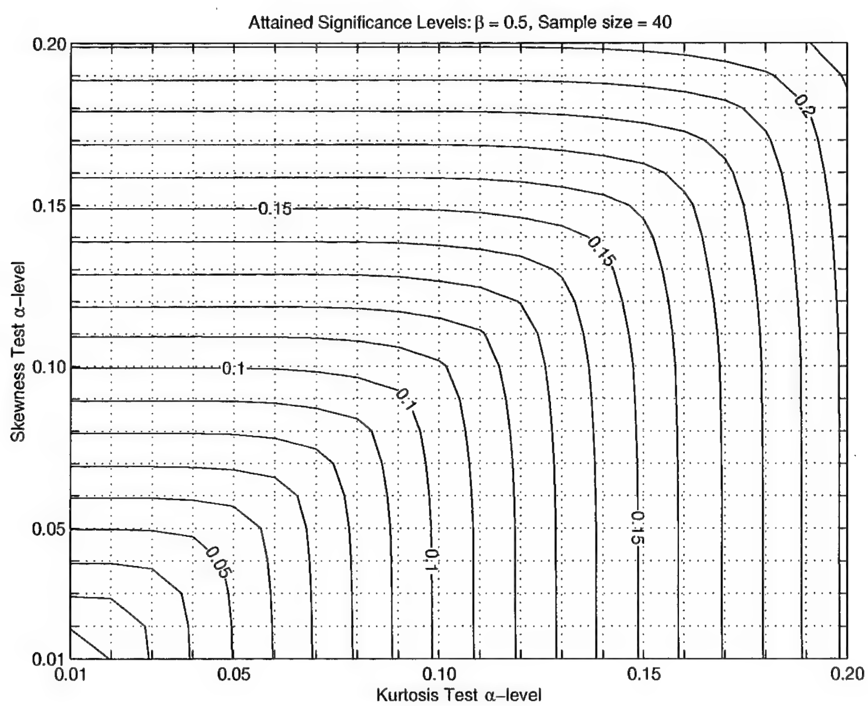
(e) Sample Size 25



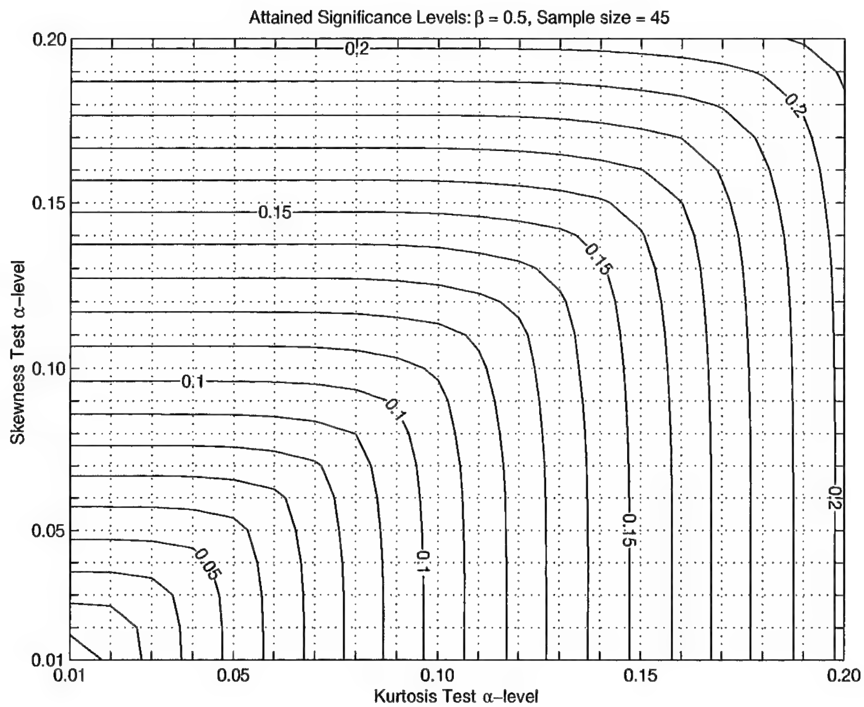
(f) Sample Size 30



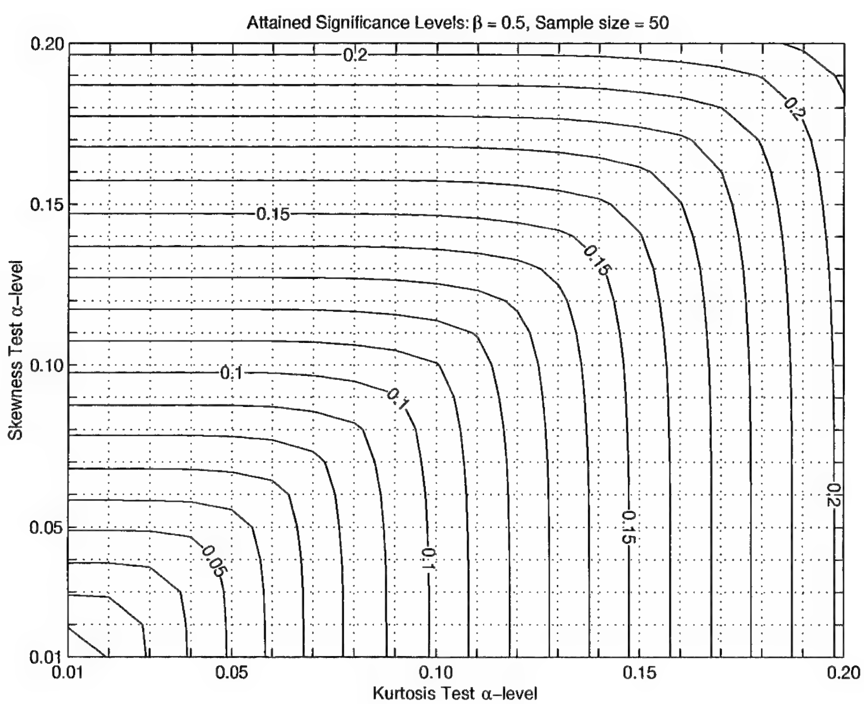
(g) Sample Size 35



(h) Sample Size 40

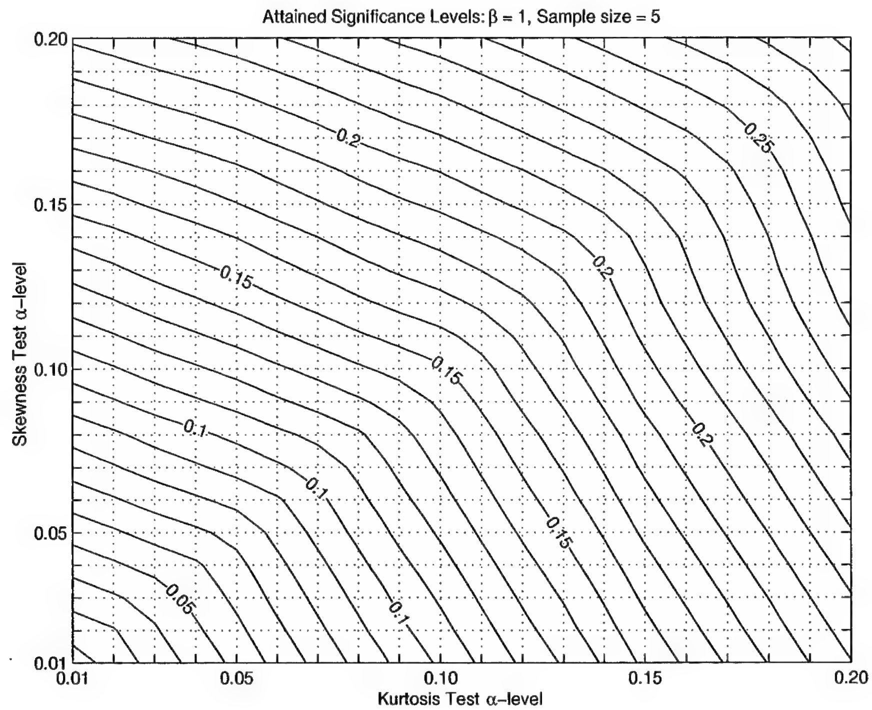


(i) Sample Size 45

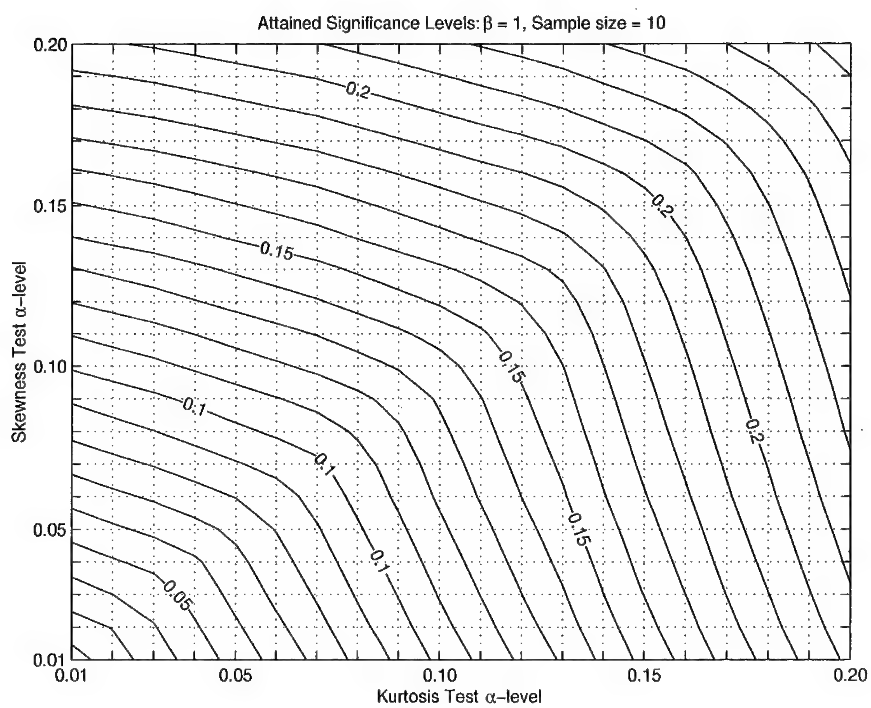


(j) Sample Size 50

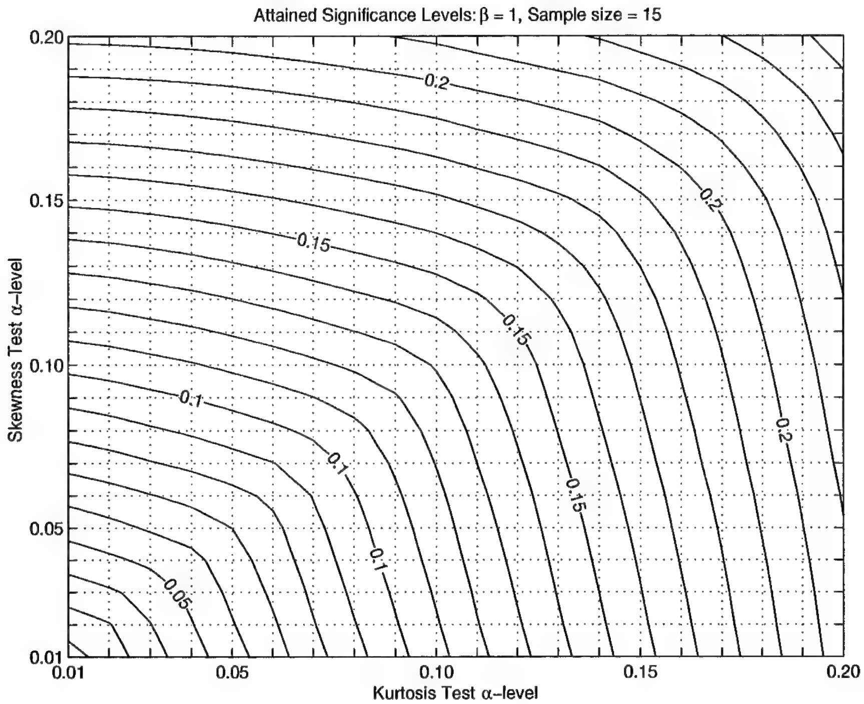
C.2 *Weibull Shape* $\beta = 1$



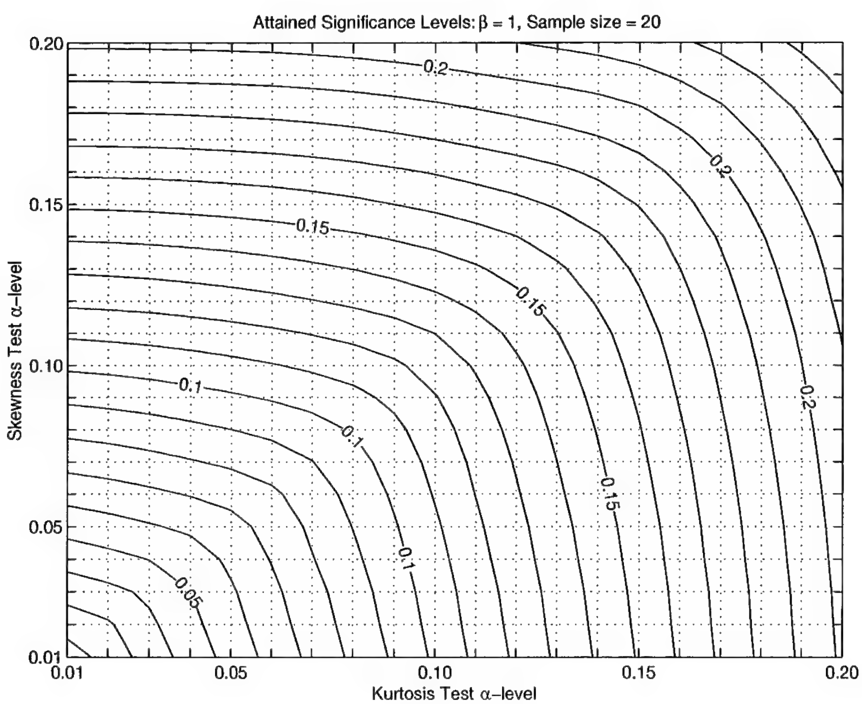
(a) Sample Size 5



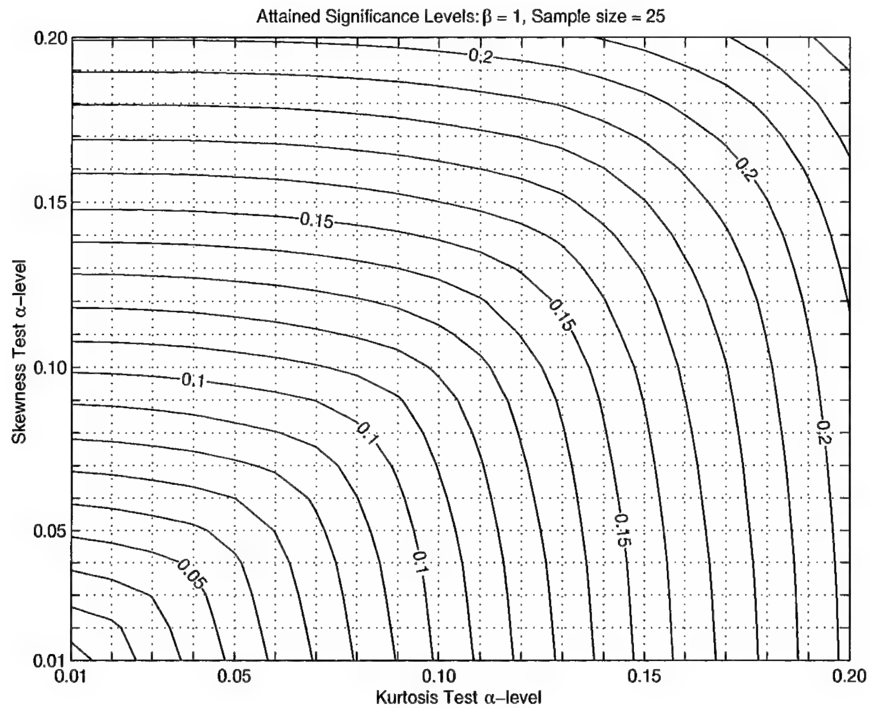
(b) Sample Size 10



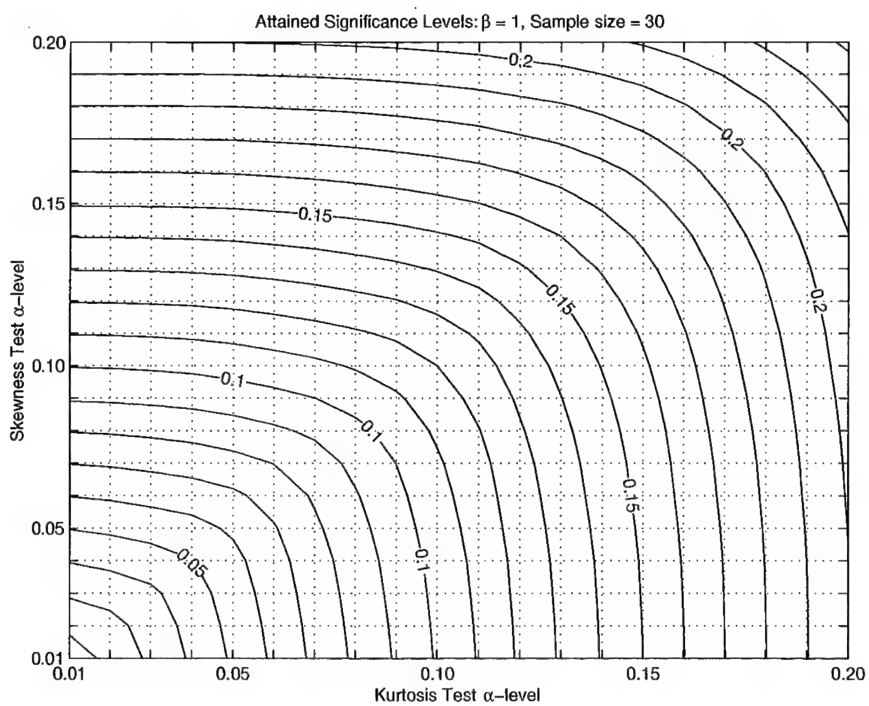
(c) Sample Size 15



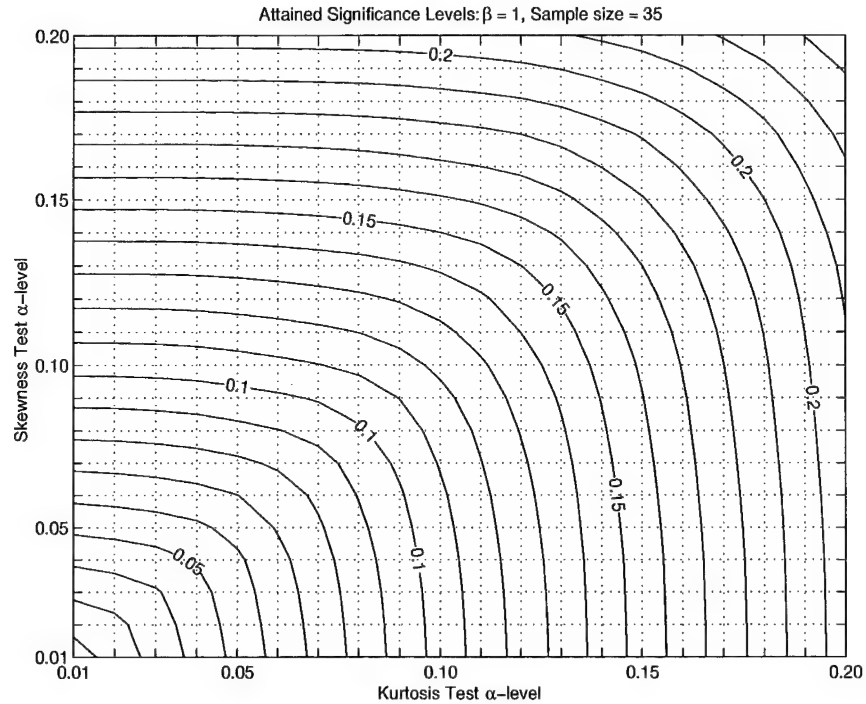
(d) Sample Size 20



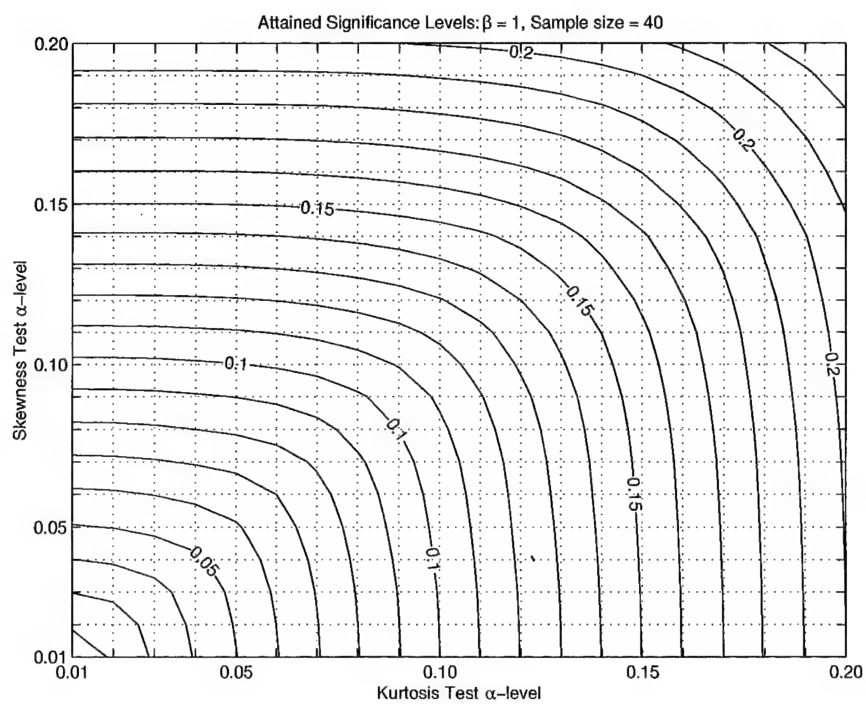
(e) Sample Size 25



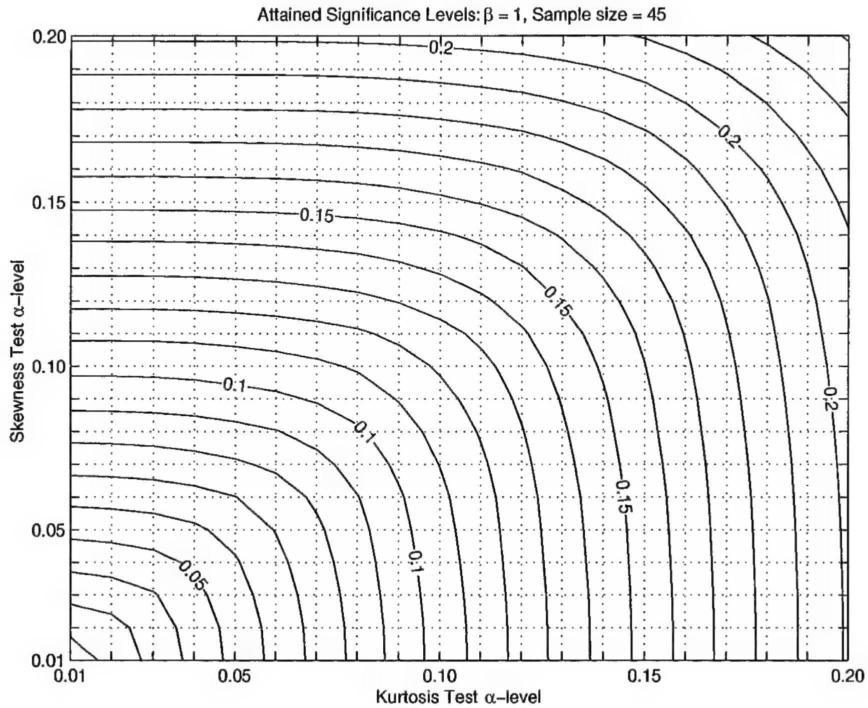
(f) Sample Size 30



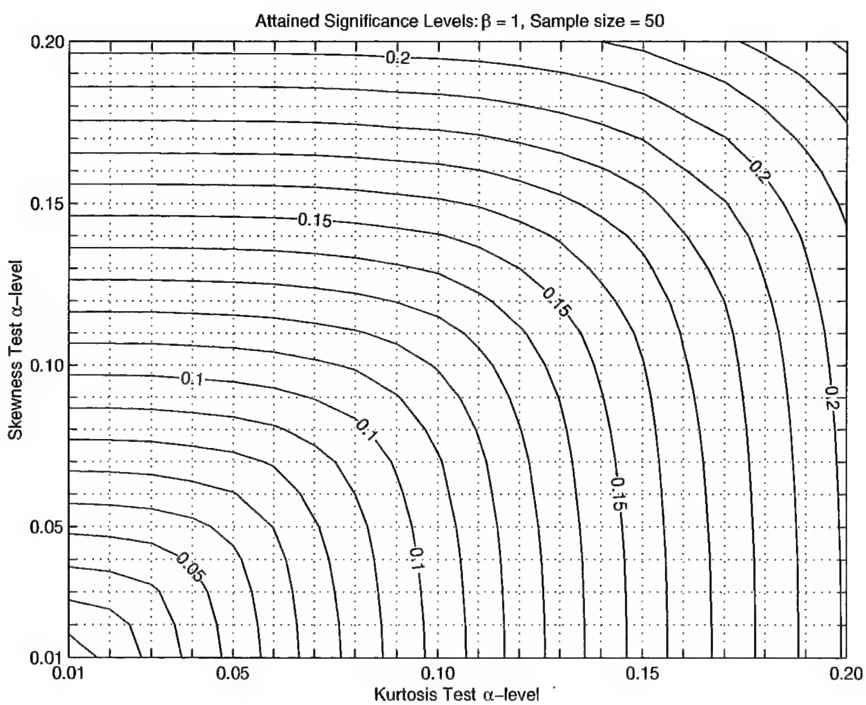
(g) Sample Size 35



(h) Sample Size 40

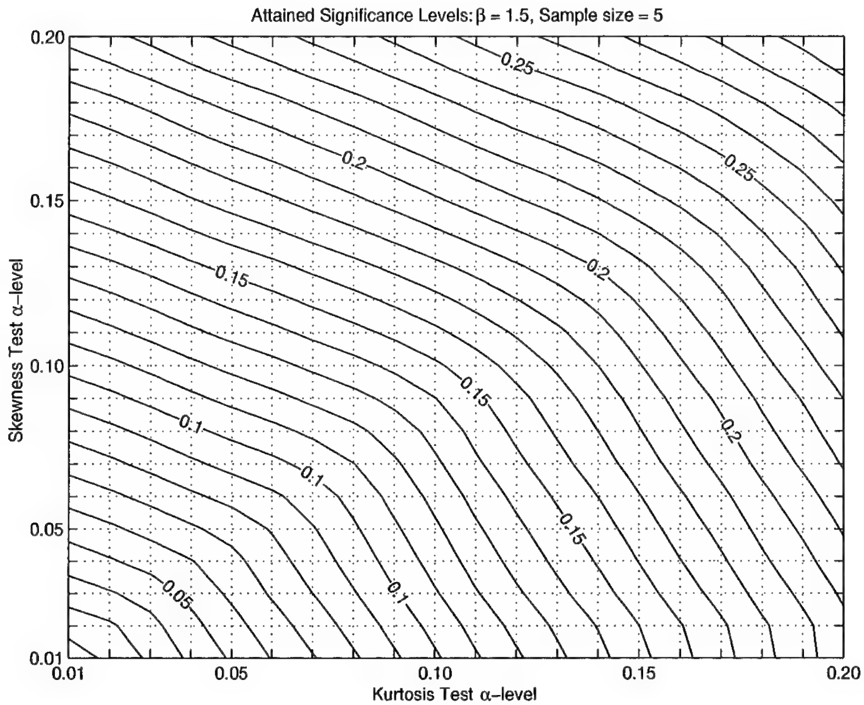


(i) Sample Size 45

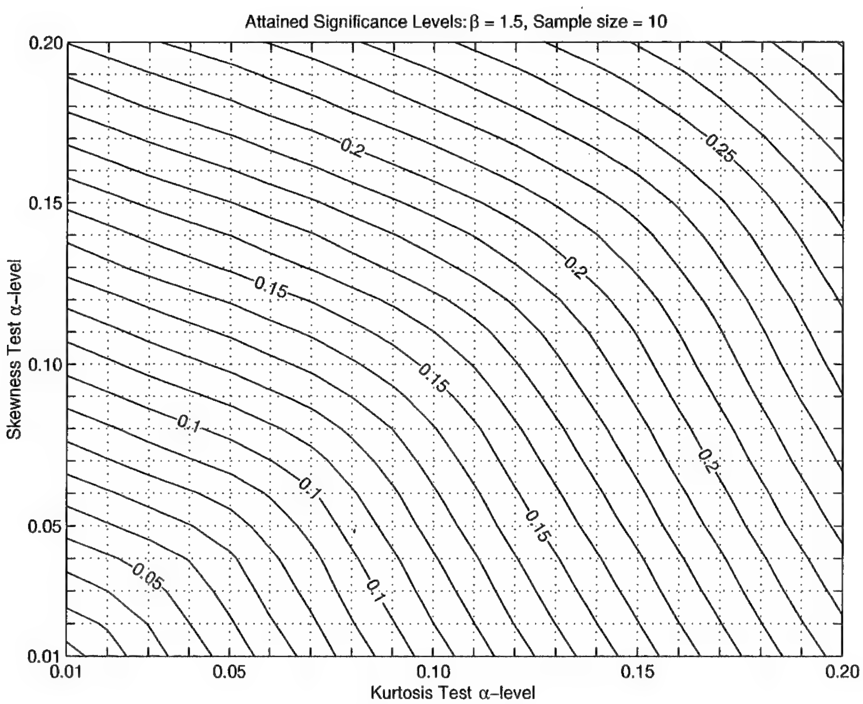


(j) Sample Size 50

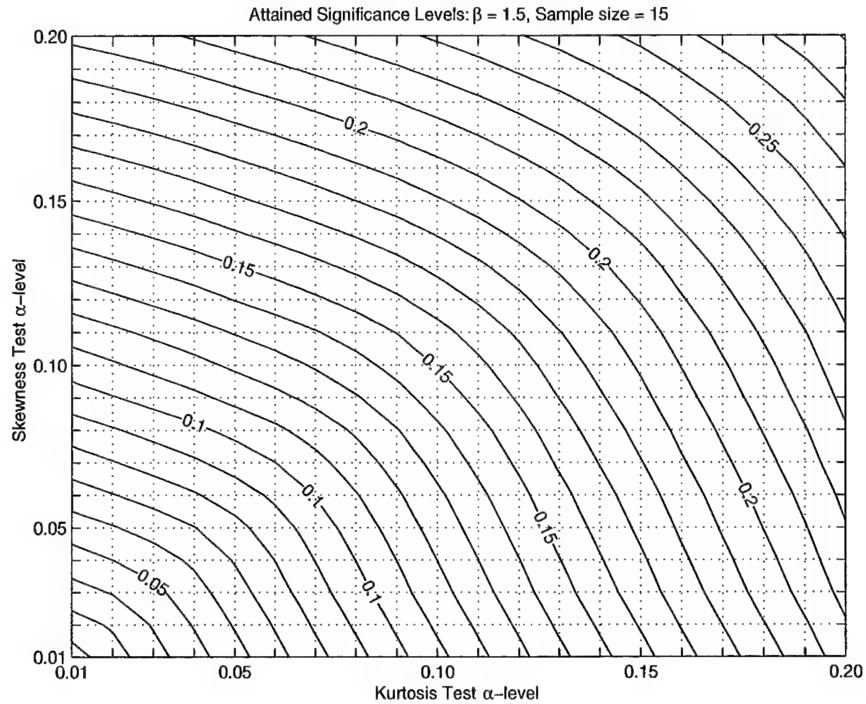
C.3 Weibull Shape $\beta = 1.5$



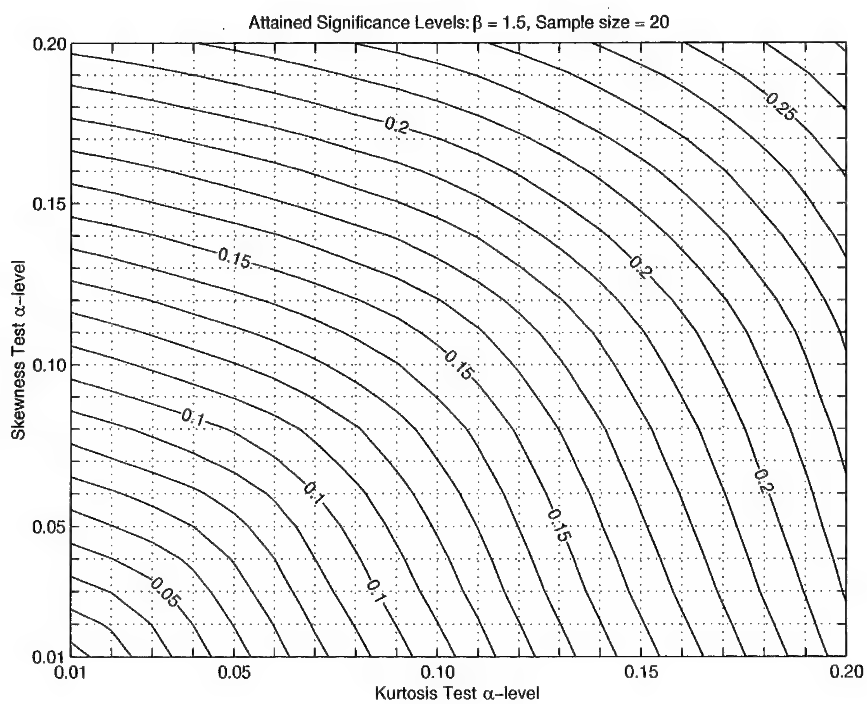
(a) Sample Size 5



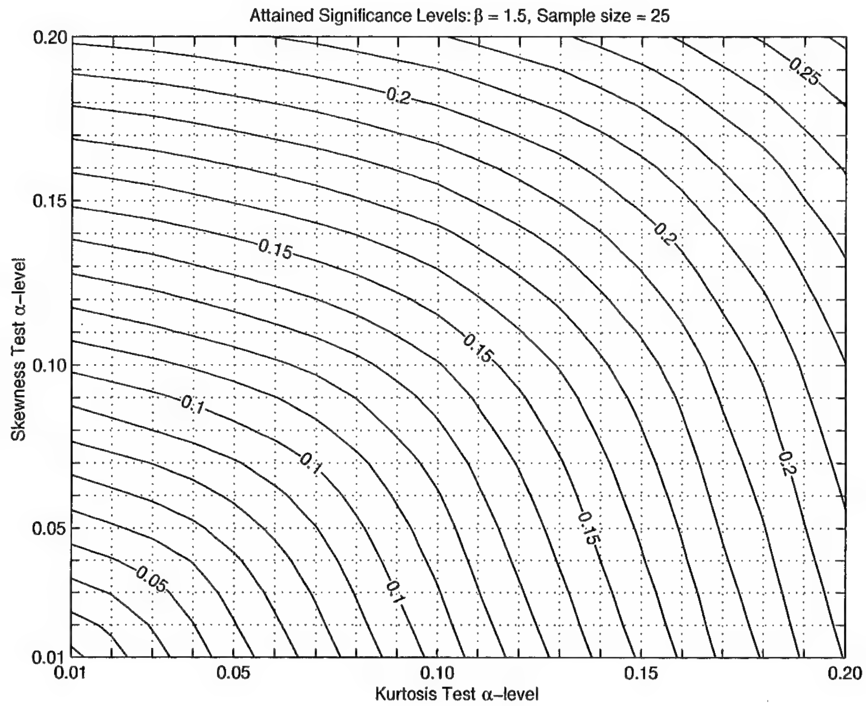
(b) Sample Size 10



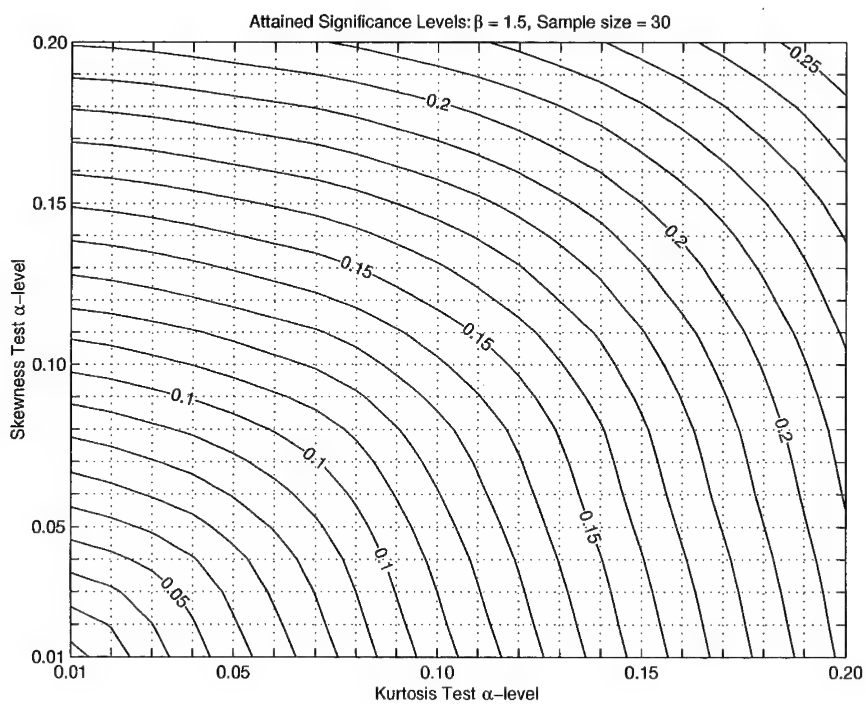
(c) Sample Size 15



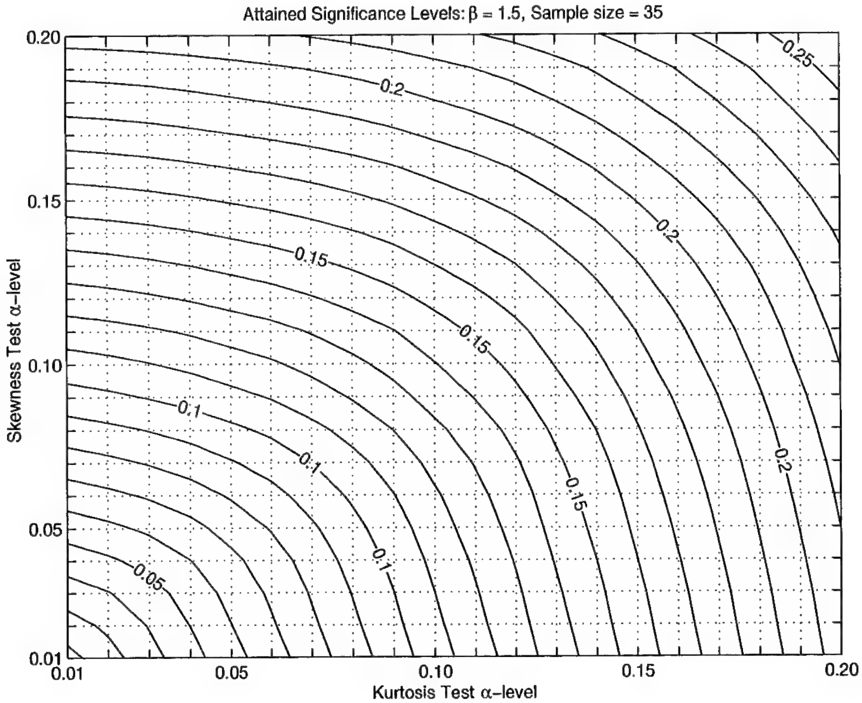
(d) Sample Size 20



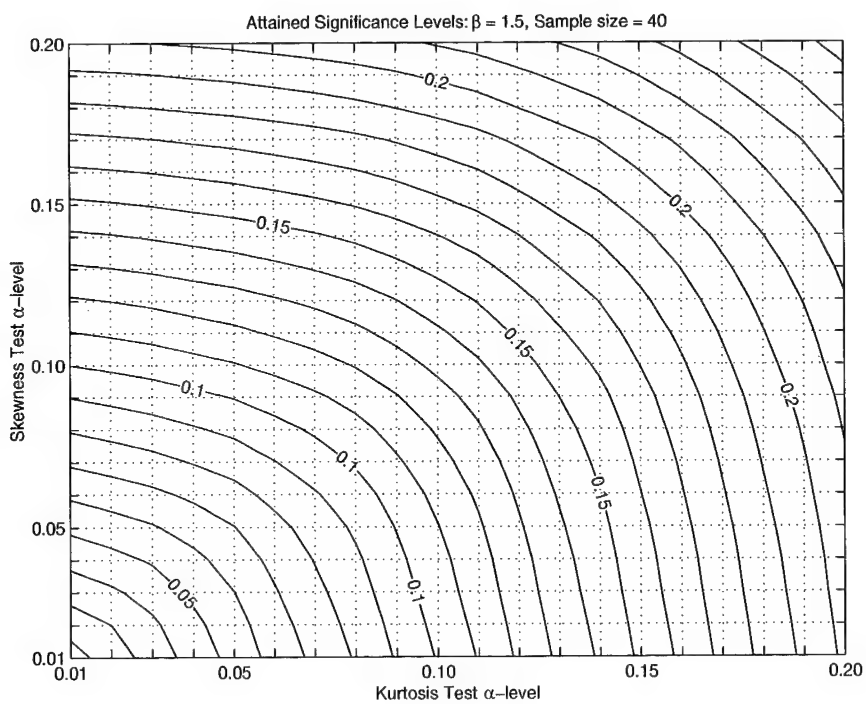
(e) Sample Size 25



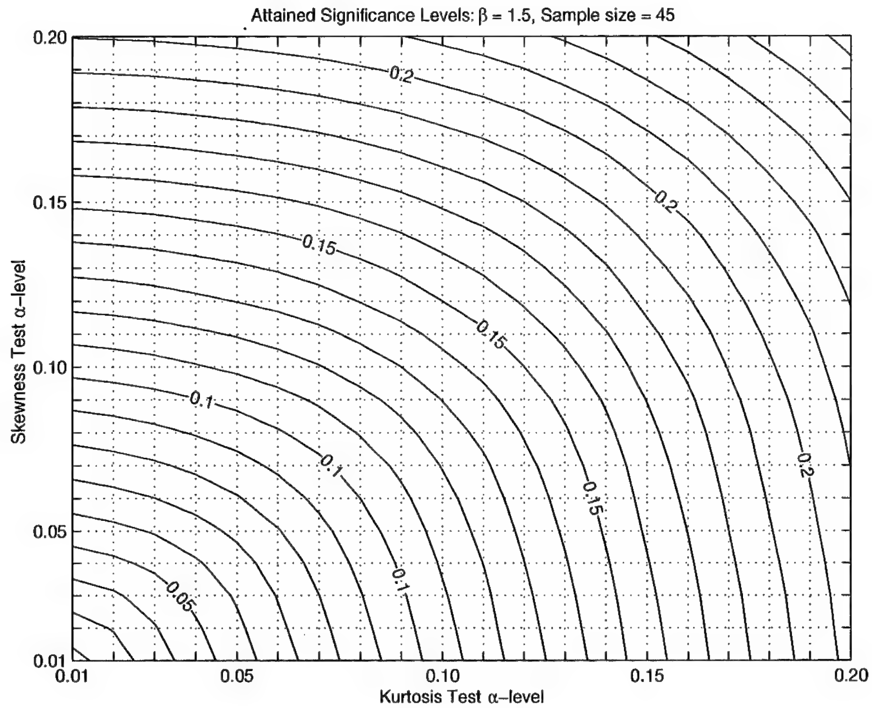
(f) Sample Size 30



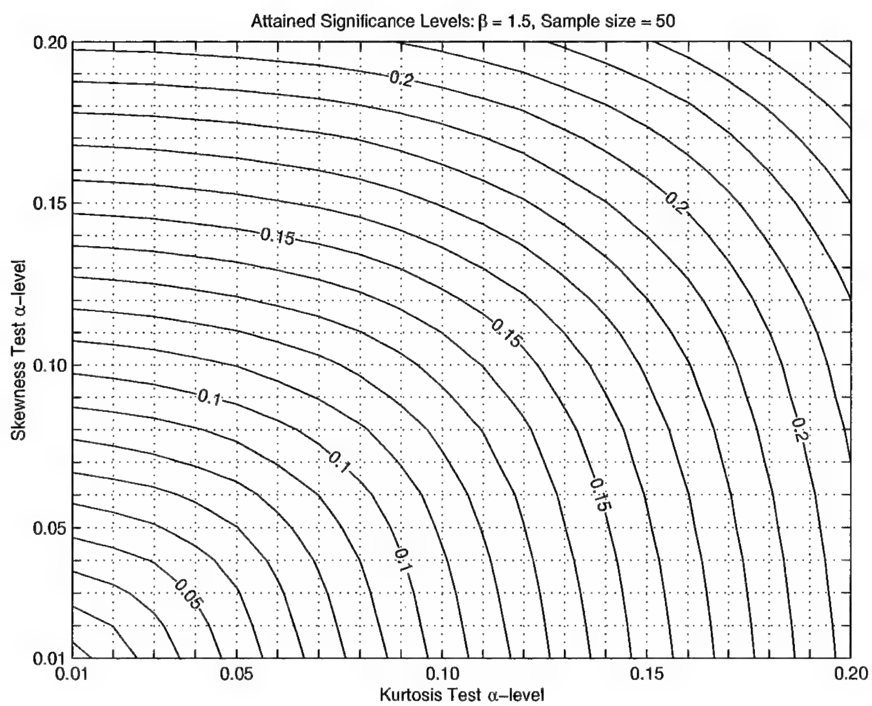
(g) Sample Size 35



(h) Sample Size 40

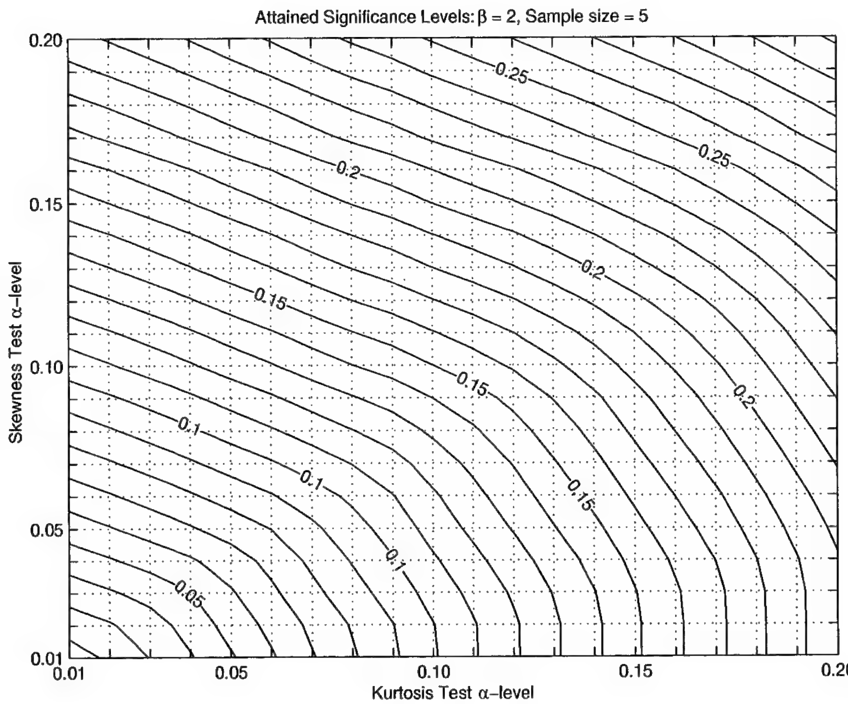


(i) Sample Size 45

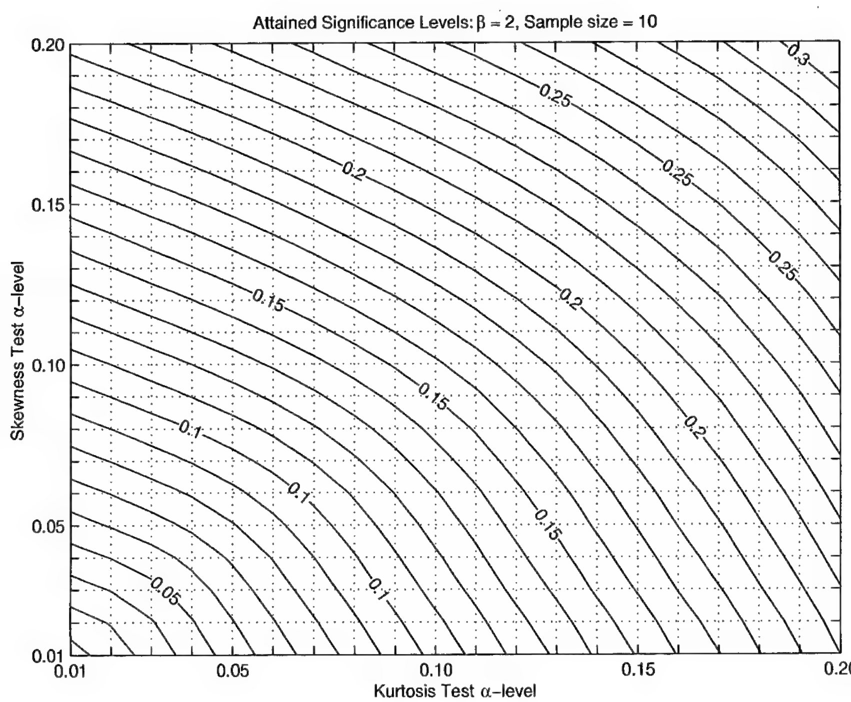


(j) Sample Size 50

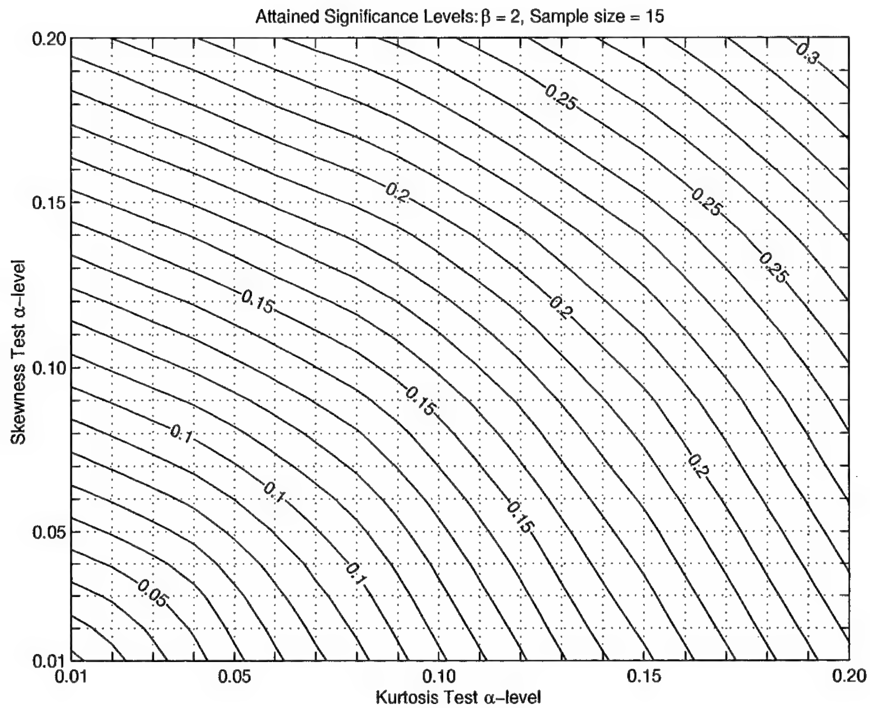
C.4 *Weibull Shape* $\beta = 2$



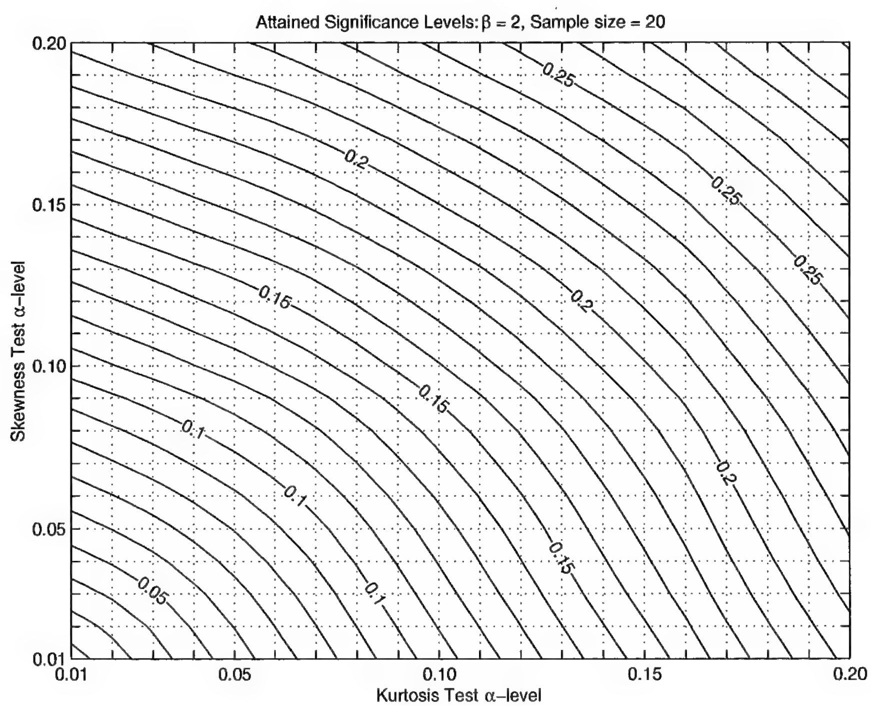
(a) Sample Size 5



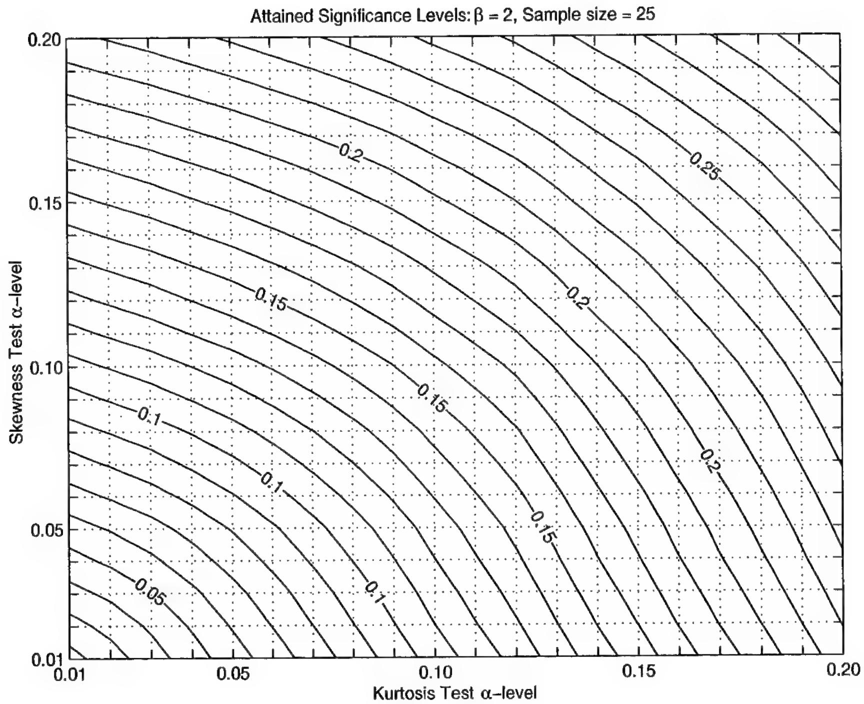
(b) Sample Size 10



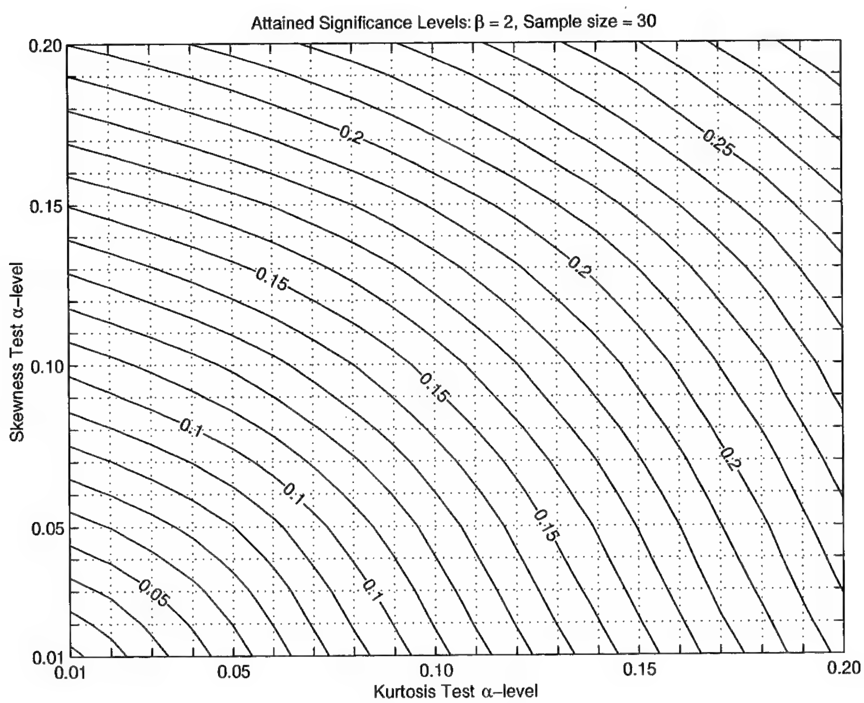
(c) Sample Size 15



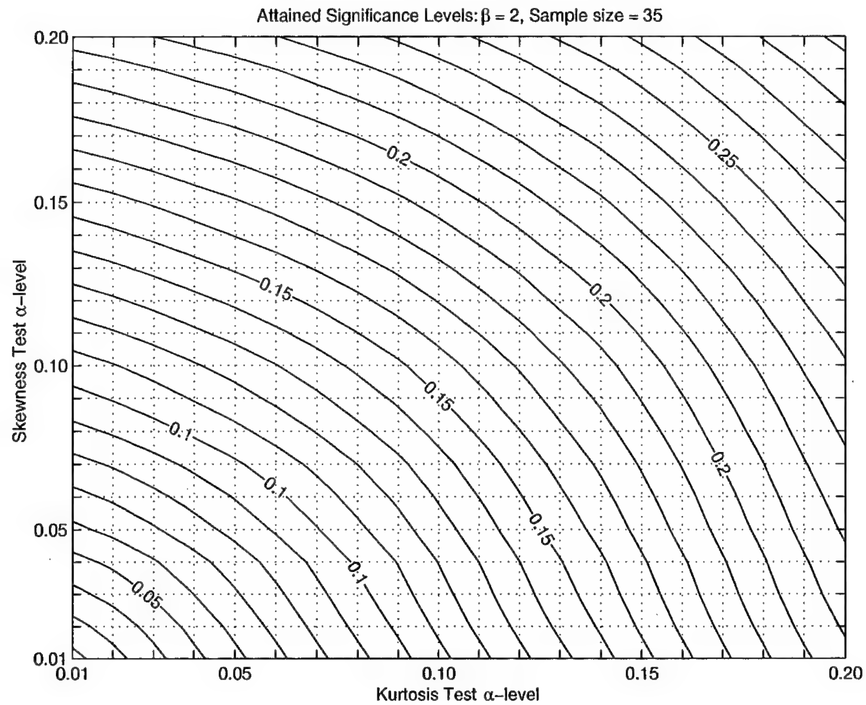
(d) Sample Size 20



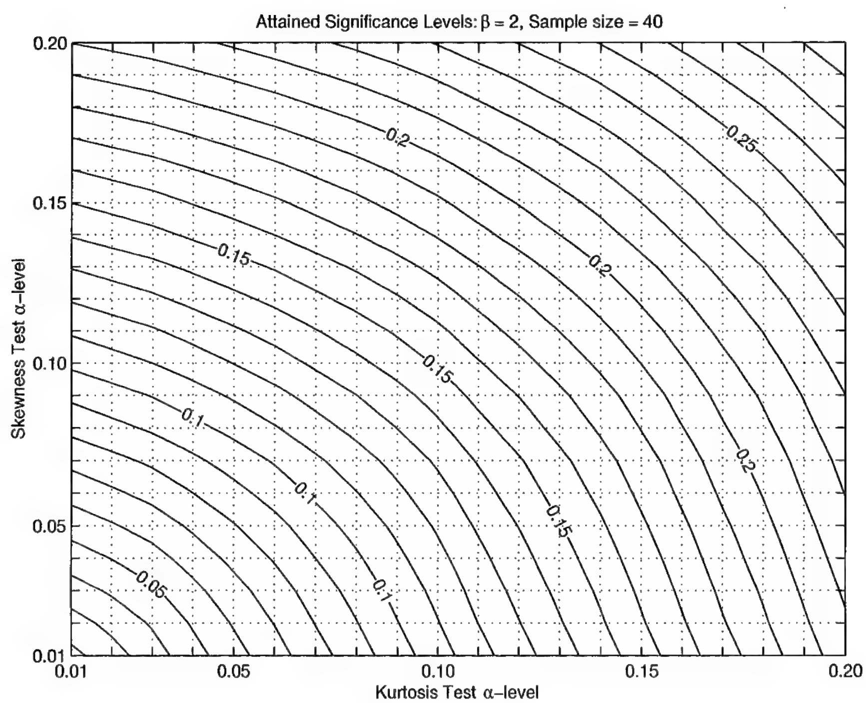
(e) Sample Size 25



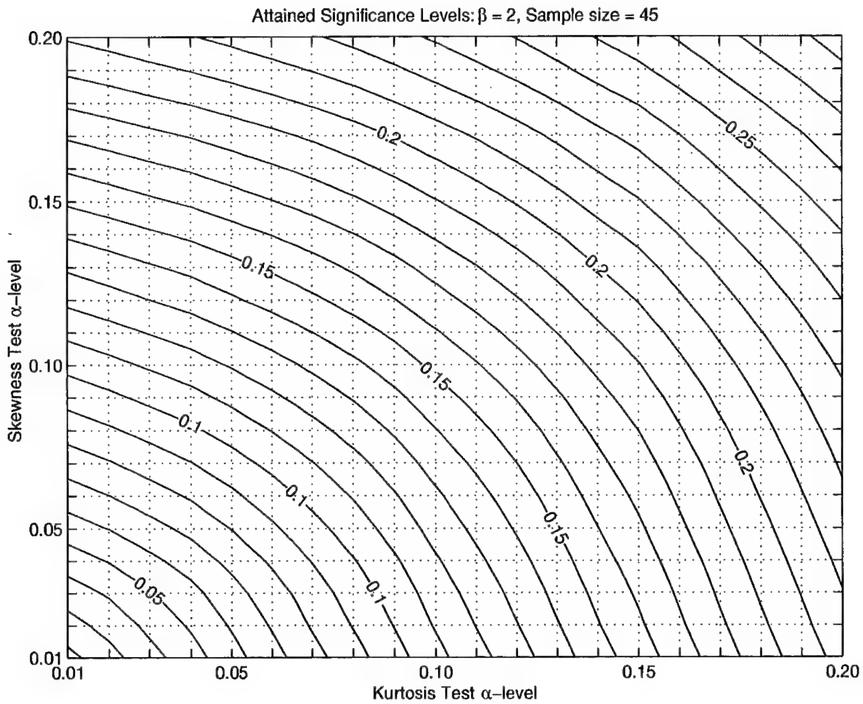
(f) Sample Size 30



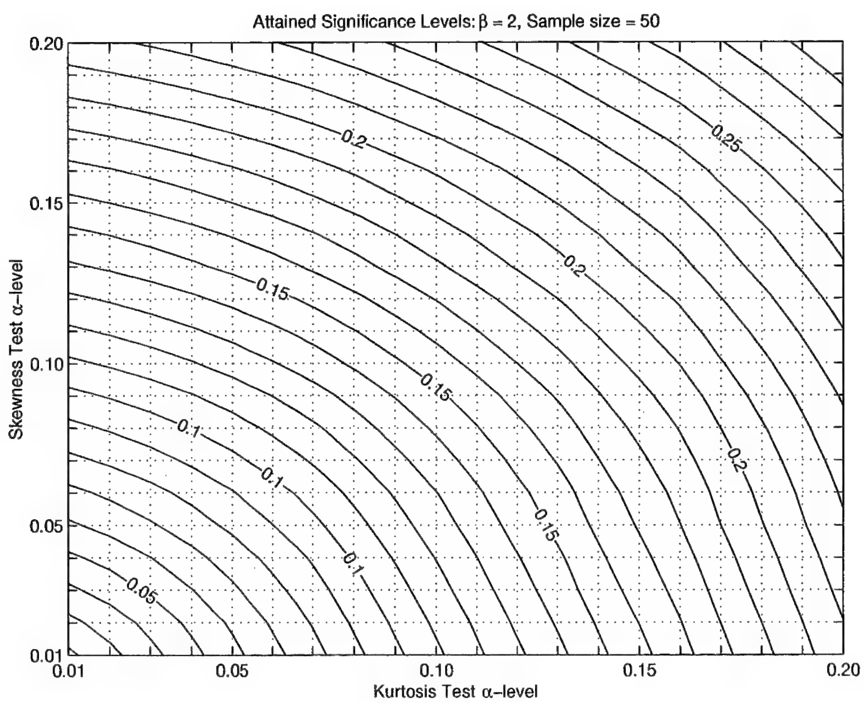
(g) Sample Size 35



(h) Sample Size 40

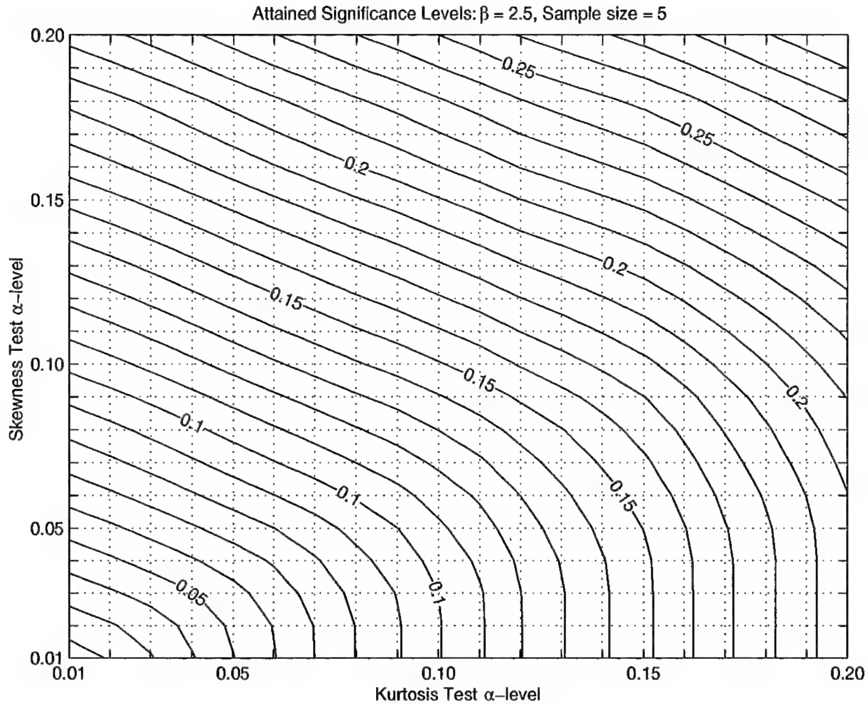


(i) Sample Size 45

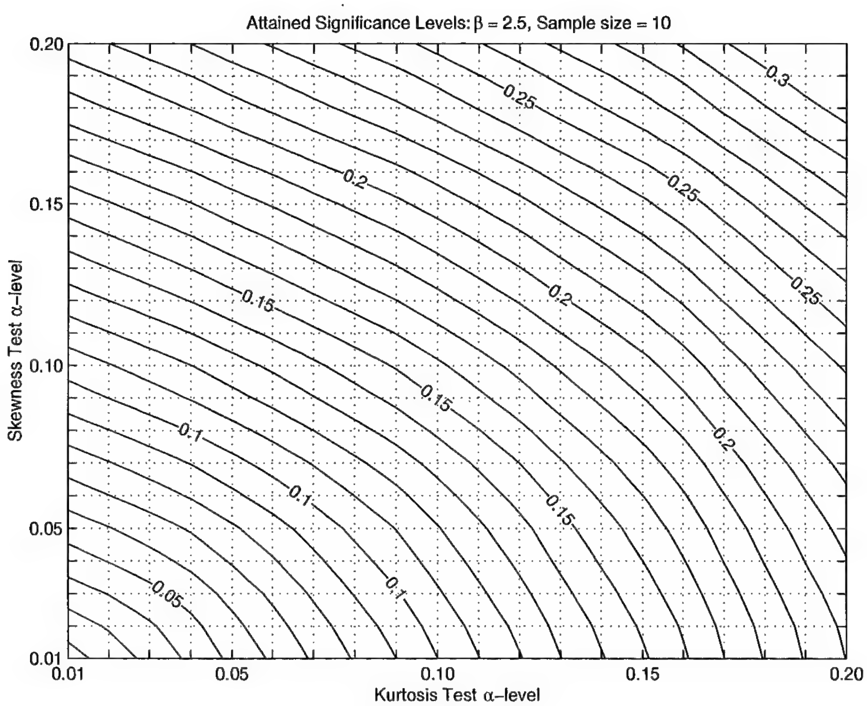


(j) Sample Size 50

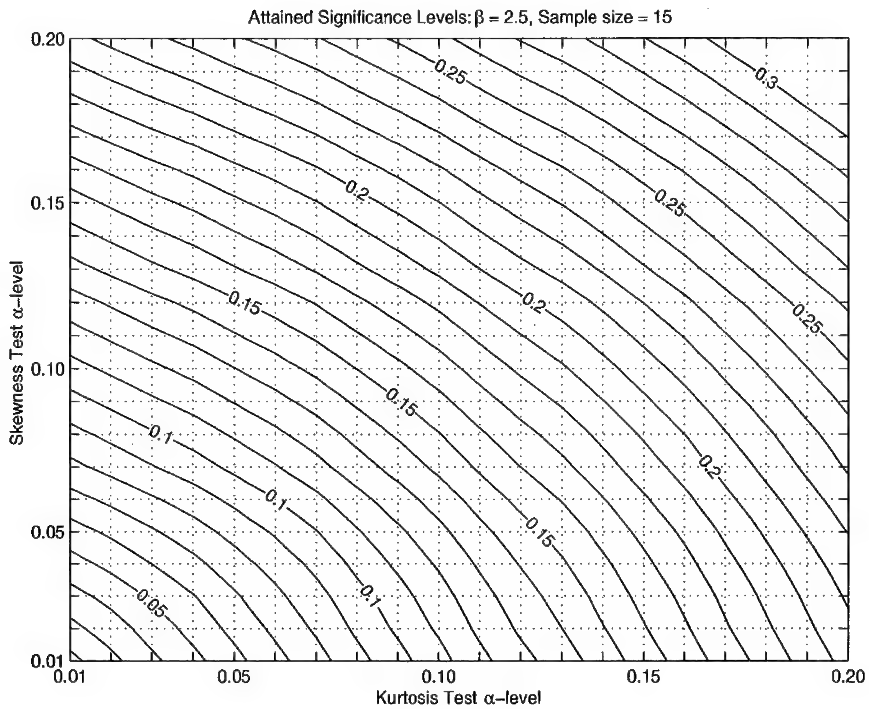
C.5 *Weibull Shape* $\beta = 2.5$



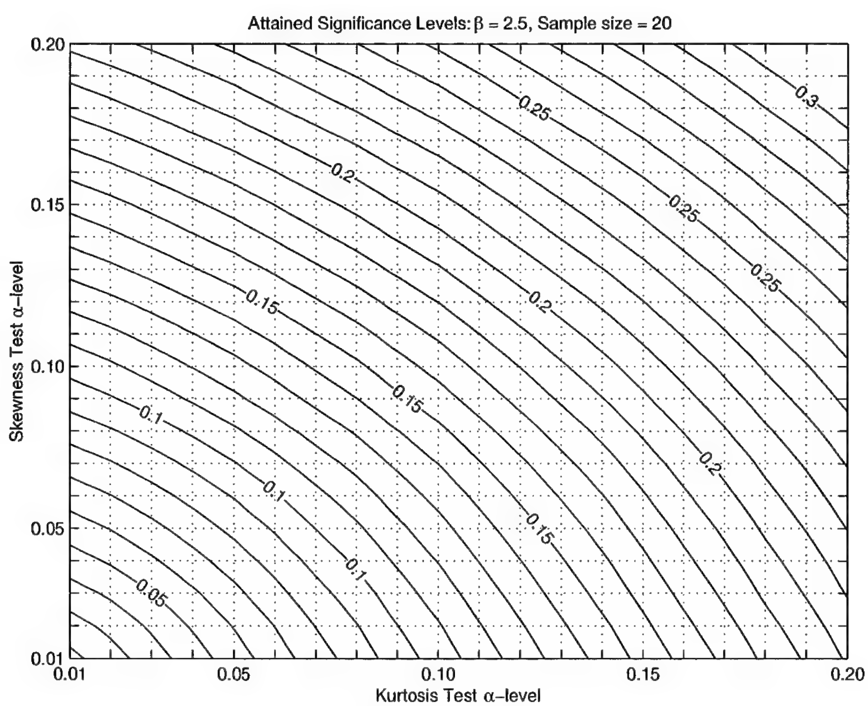
(a) Sample Size 5



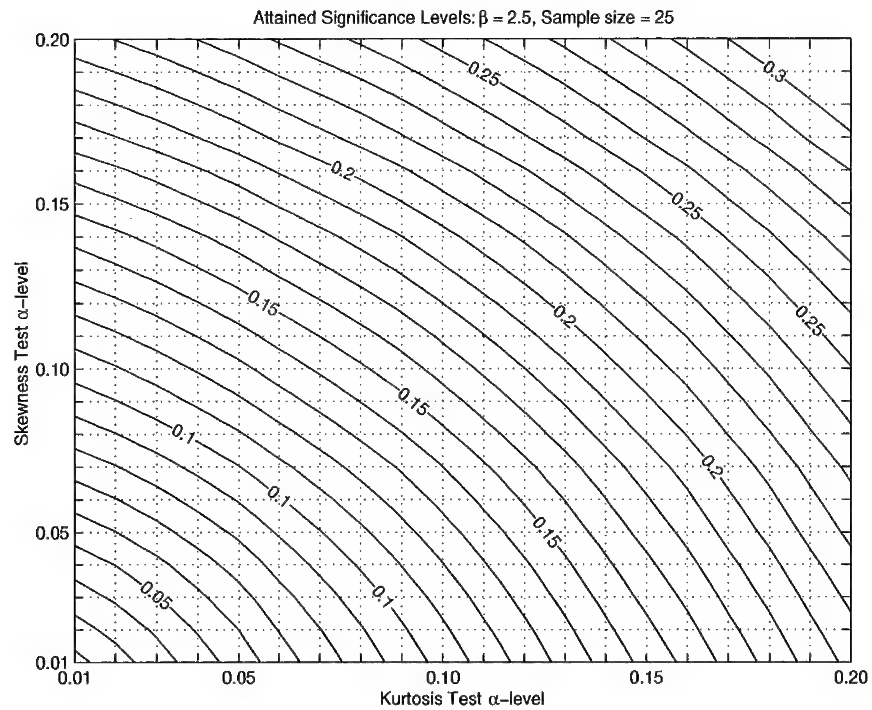
(b) Sample Size 10



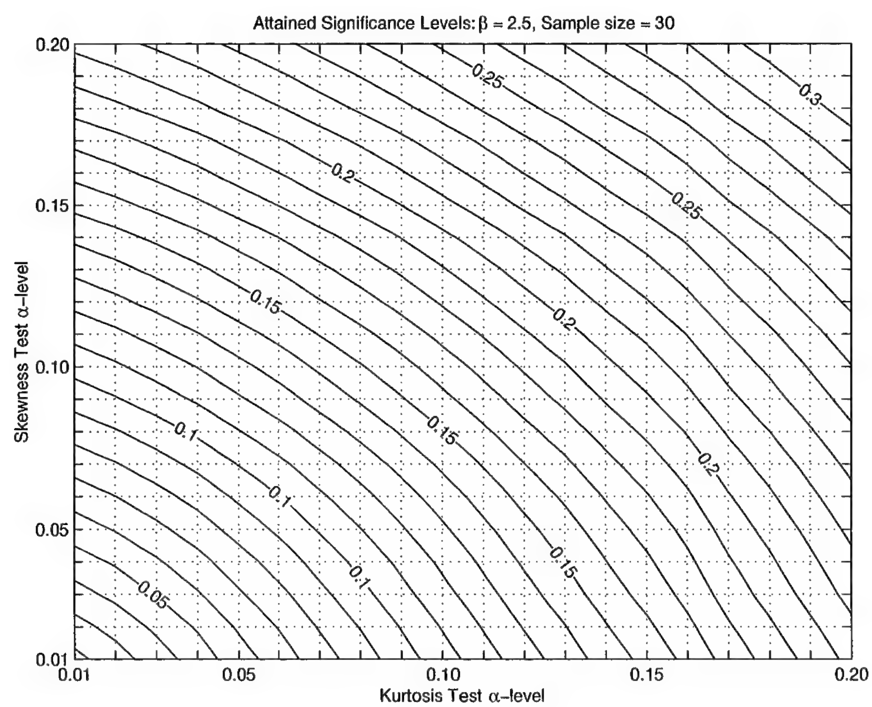
(c) Sample Size 15



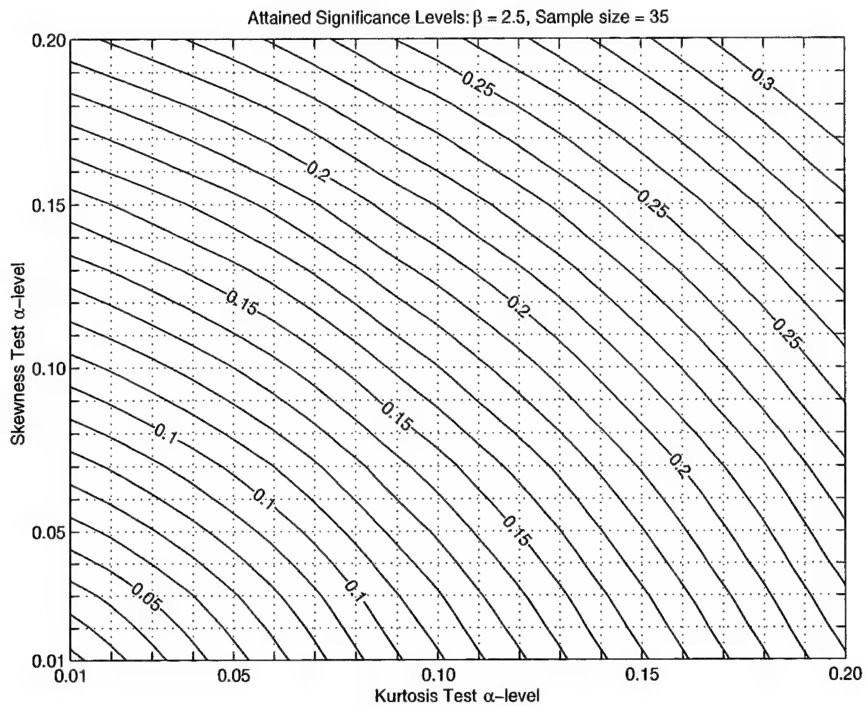
(d) Sample Size 20



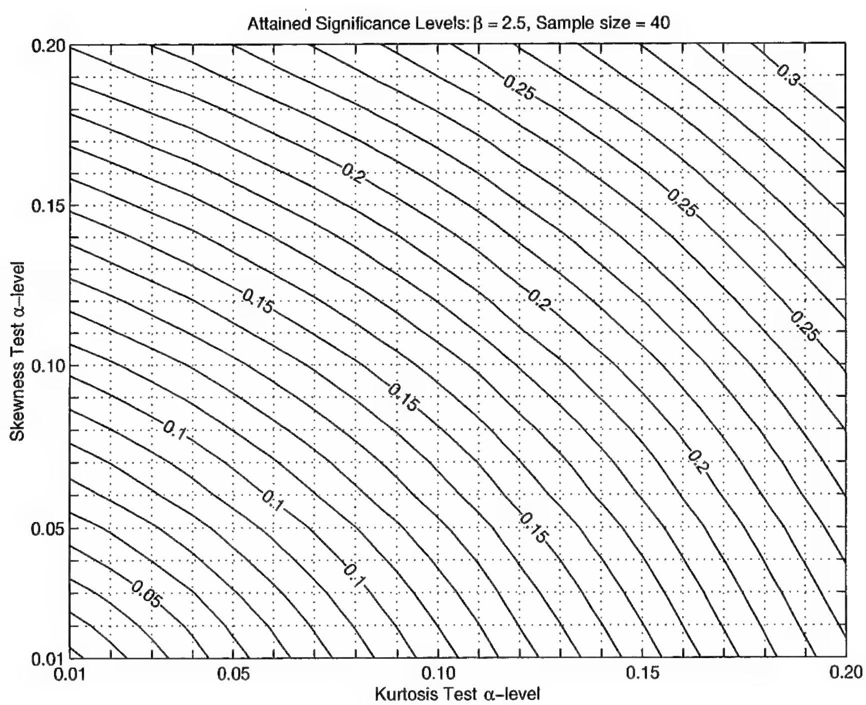
(e) Sample Size 25



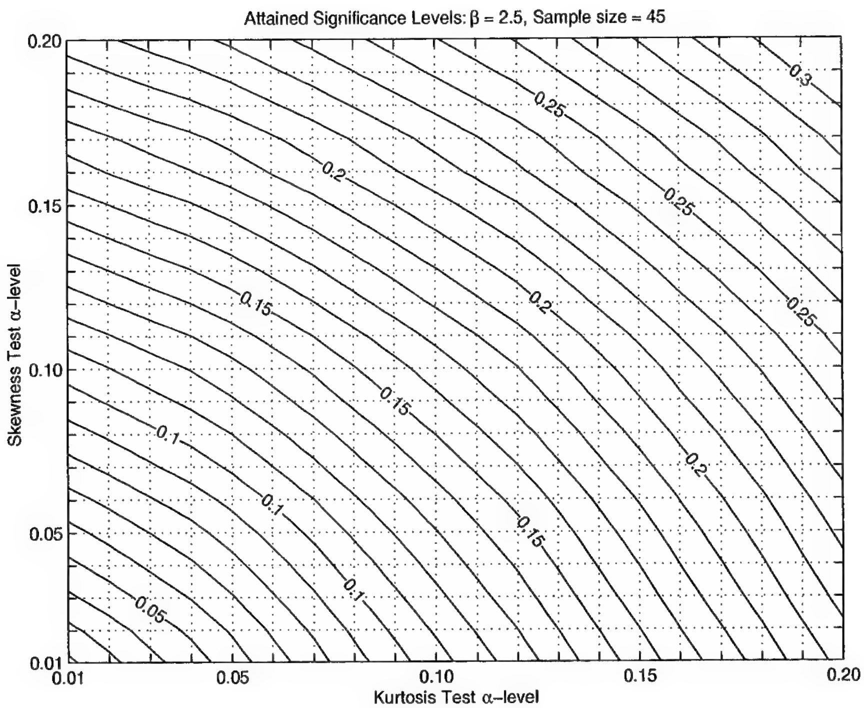
(f) Sample Size 30



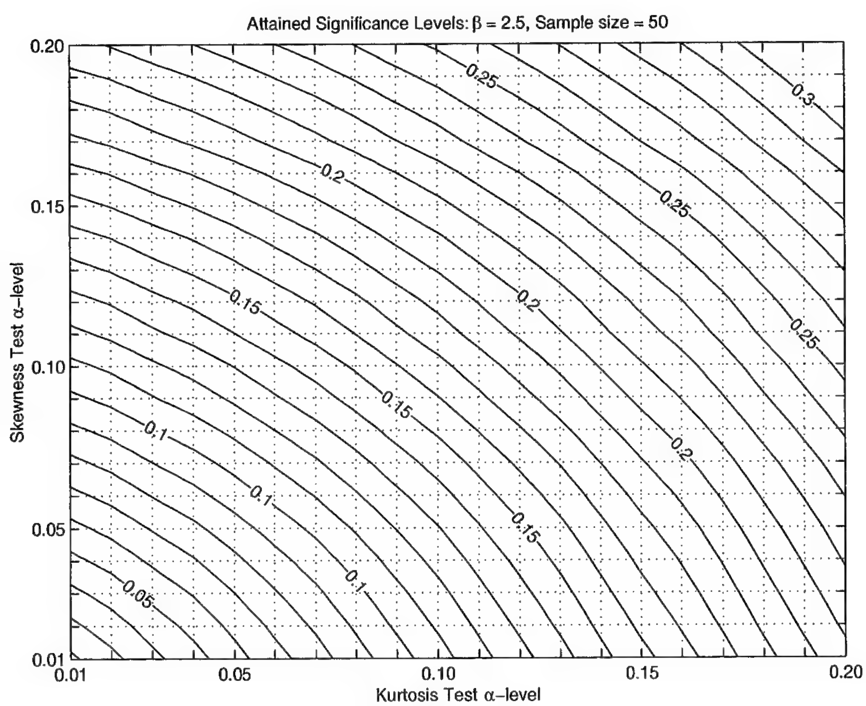
(g) Sample Size 35



(h) Sample Size 40

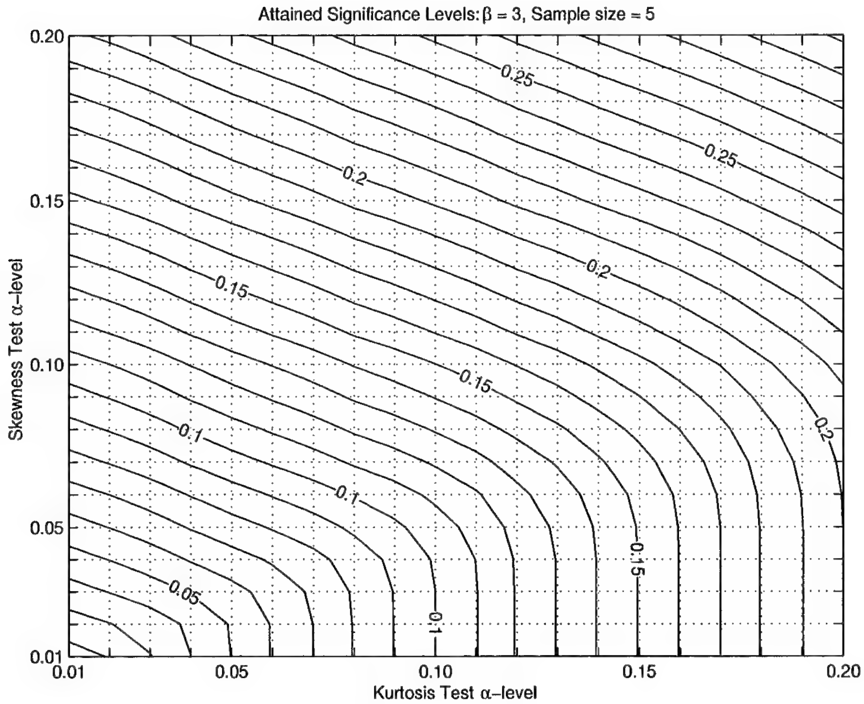


(i) Sample Size 45

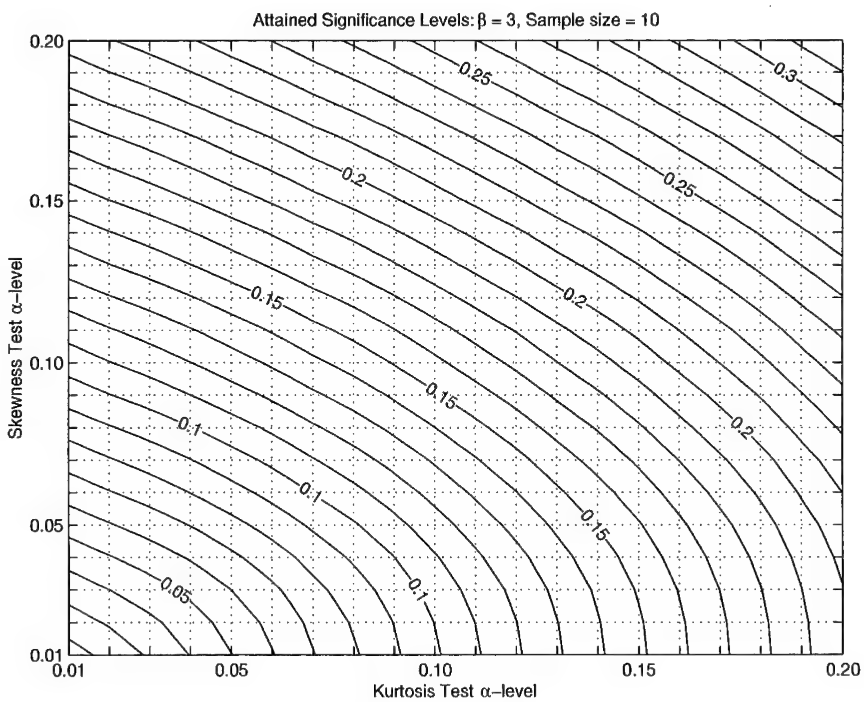


(j) Sample Size 50

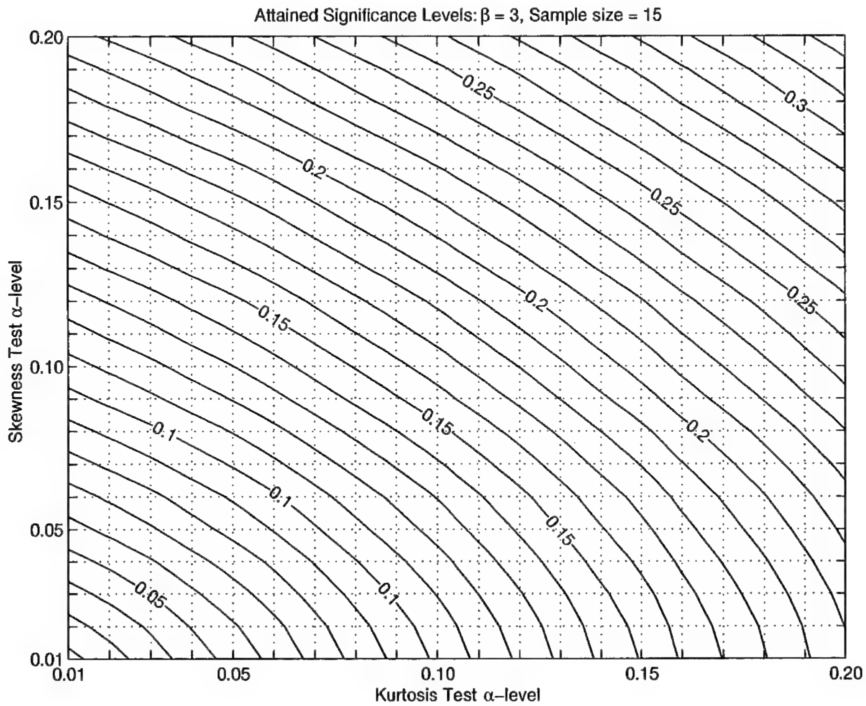
C.6 Weibull Shape $\beta = 3$



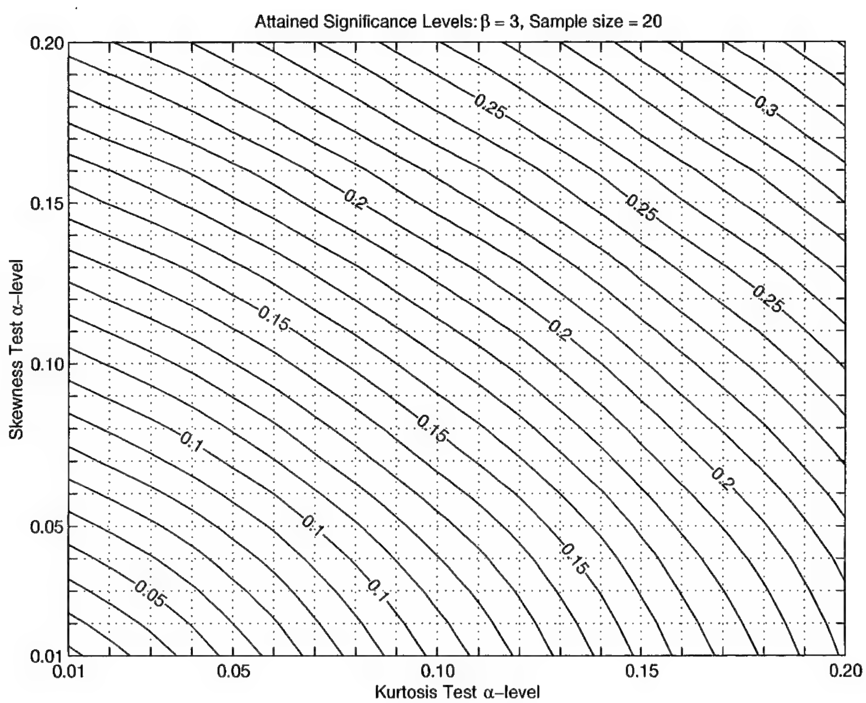
(a) Sample Size 5



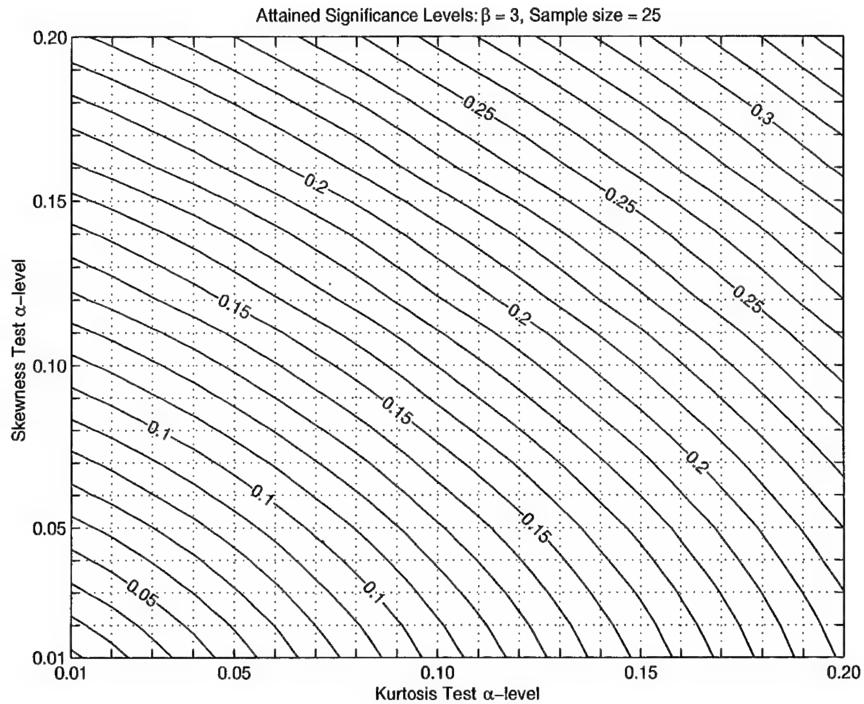
(b) Sample Size 10



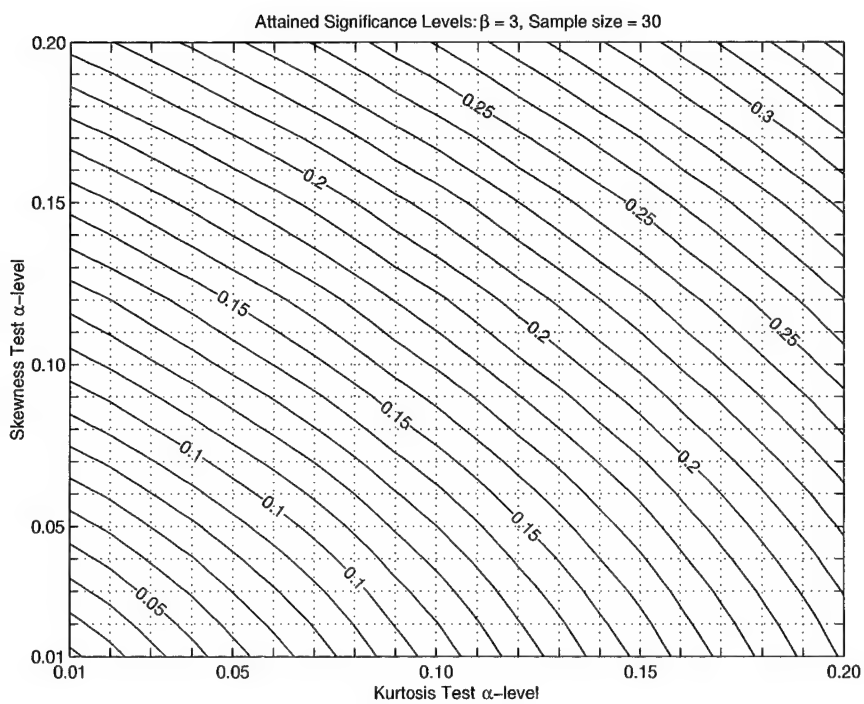
(c) Sample Size 15



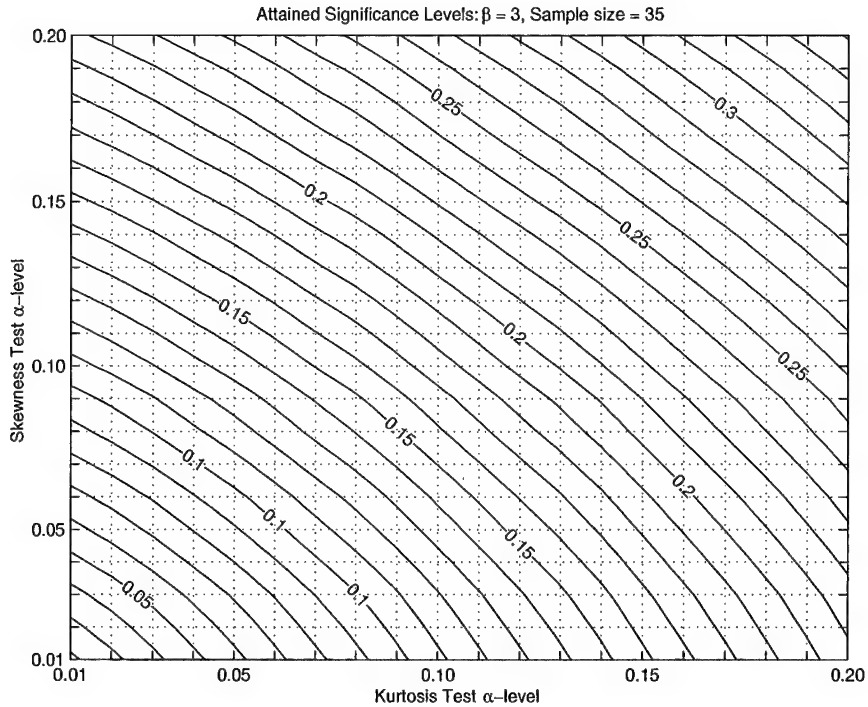
(d) Sample Size 20



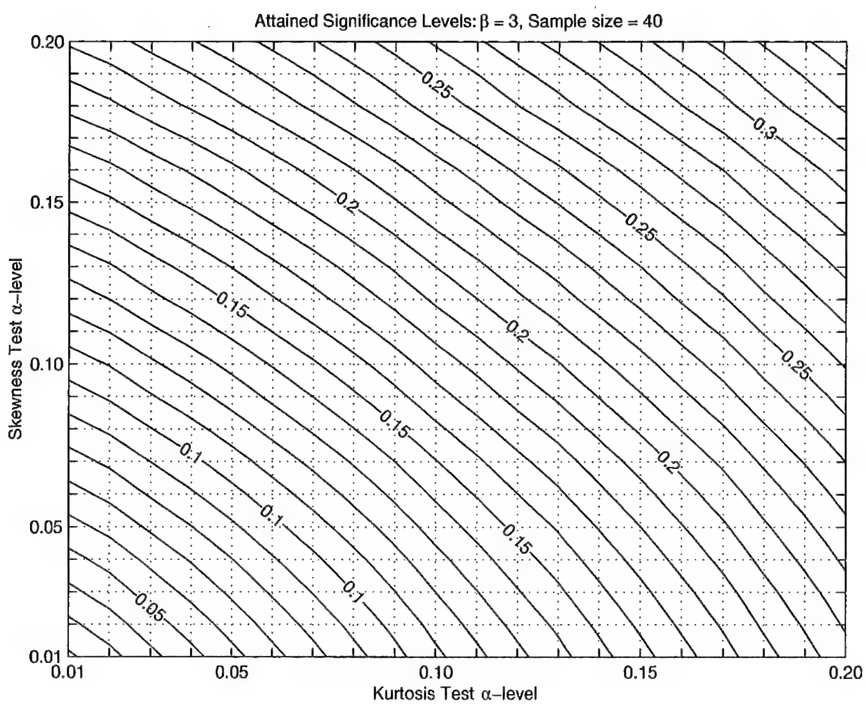
(e) Sample Size 25



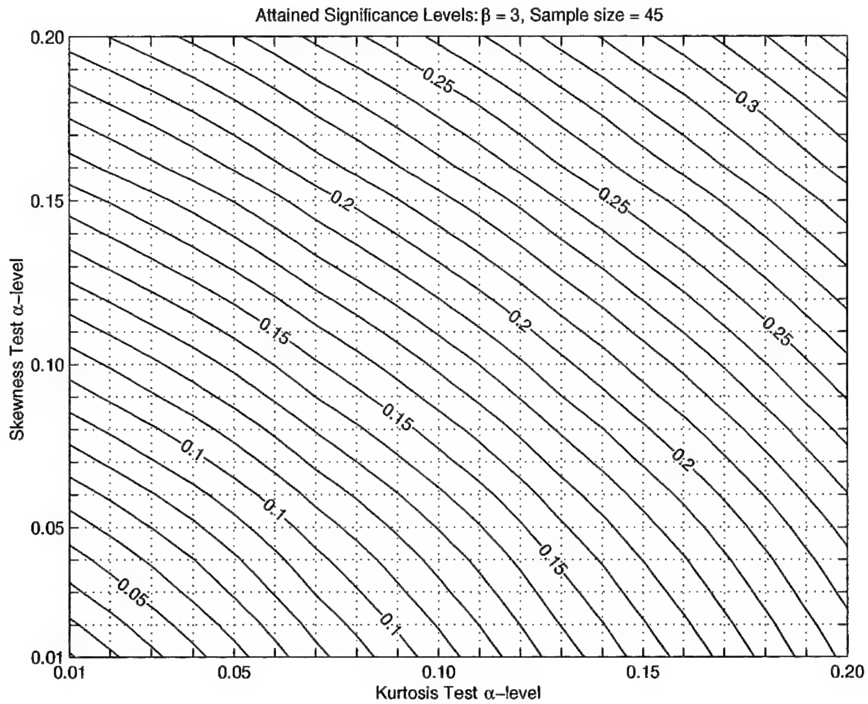
(f) Sample Size 30



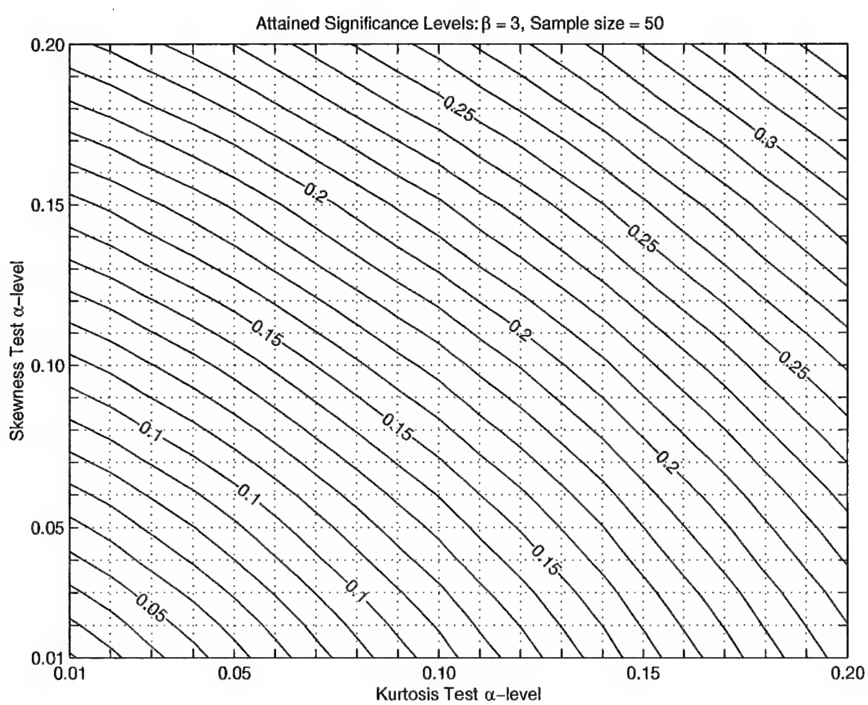
(g) Sample Size 35



(h) Sample Size 40

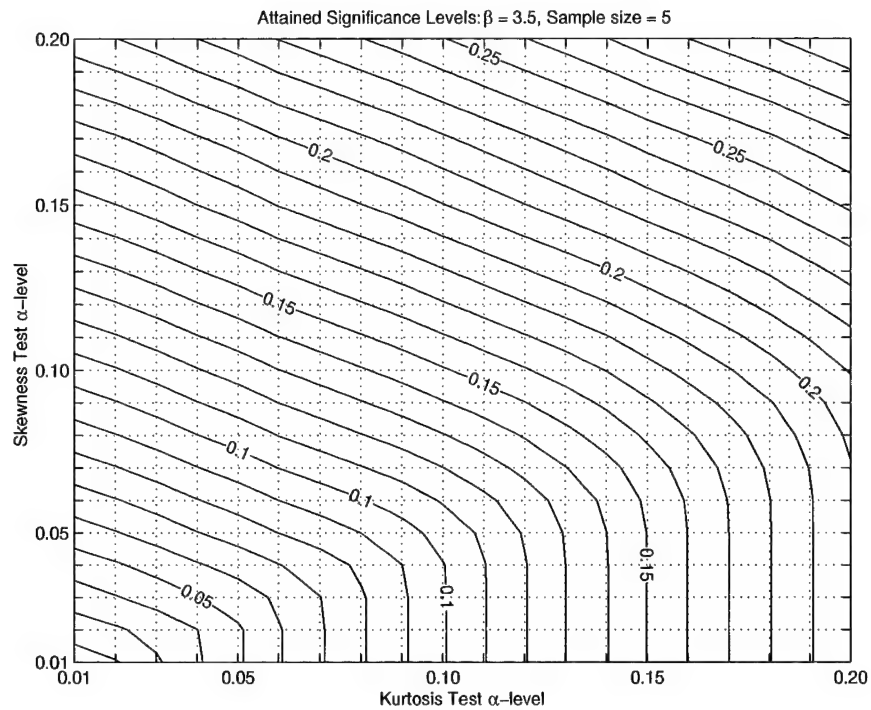


(i) Sample Size 45

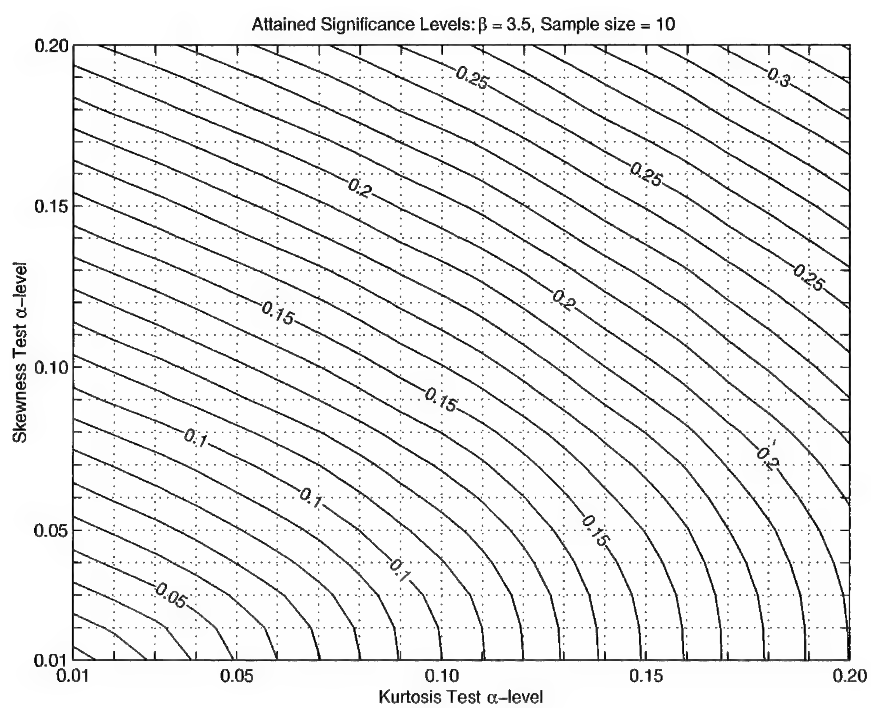


(j) Sample Size 50

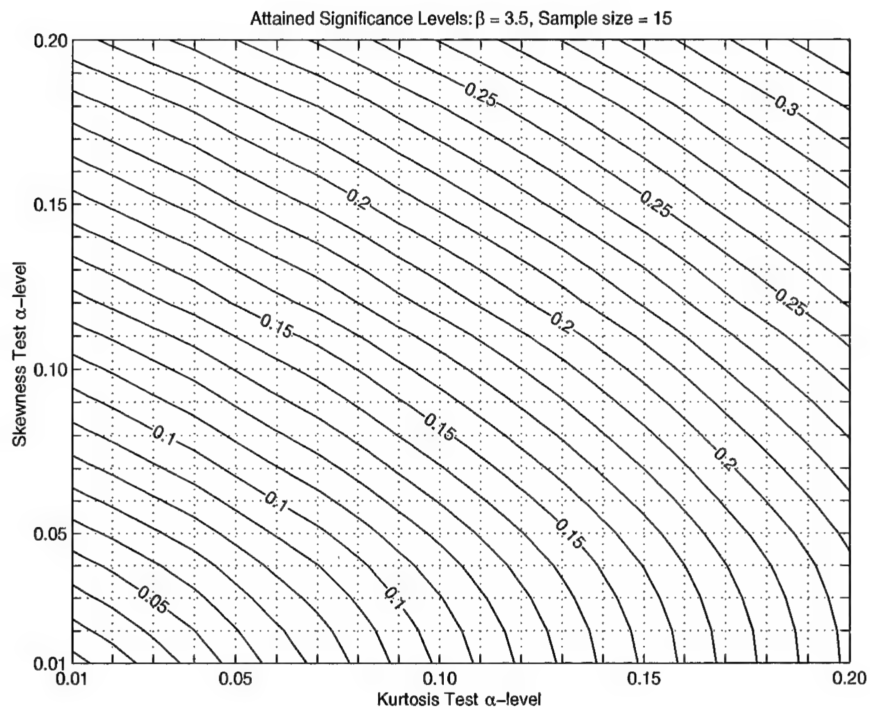
C.7 *Weibull Shape* $\beta = 3.5$



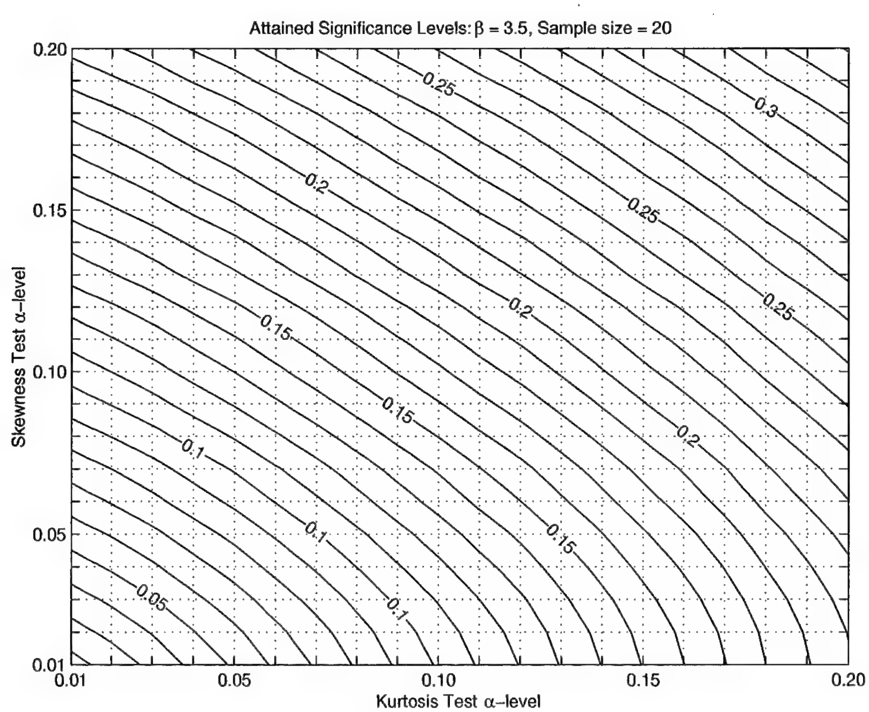
(a) Sample Size 5



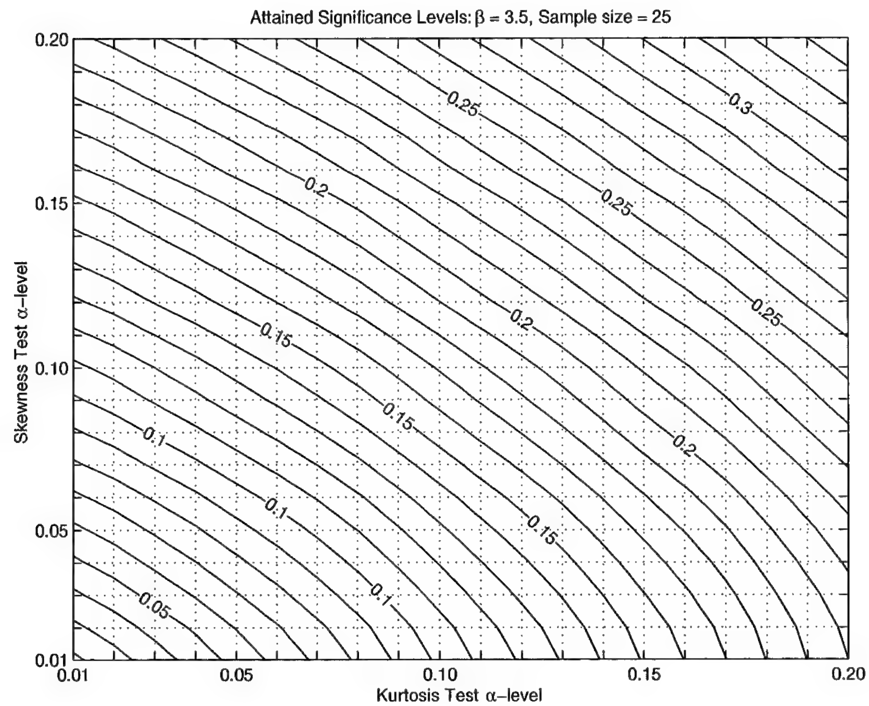
(b) Sample Size 10



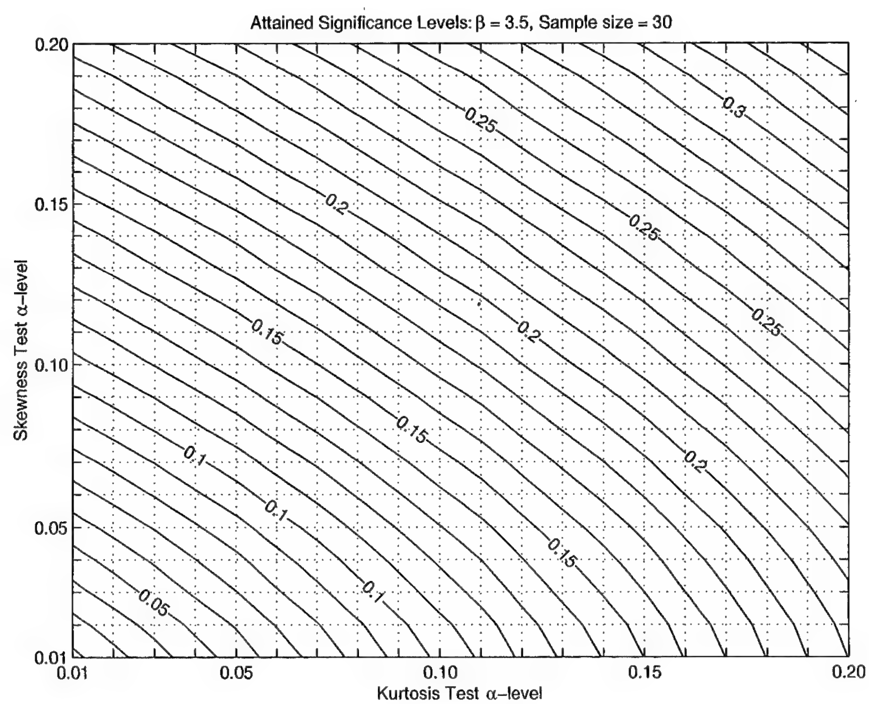
(c) Sample Size 15



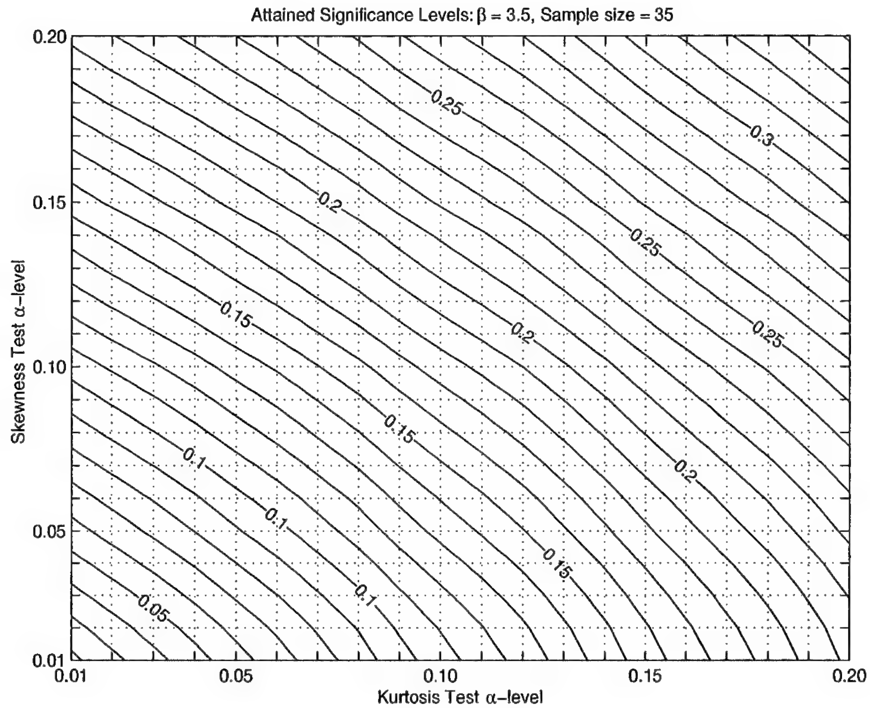
(d) Sample Size 20



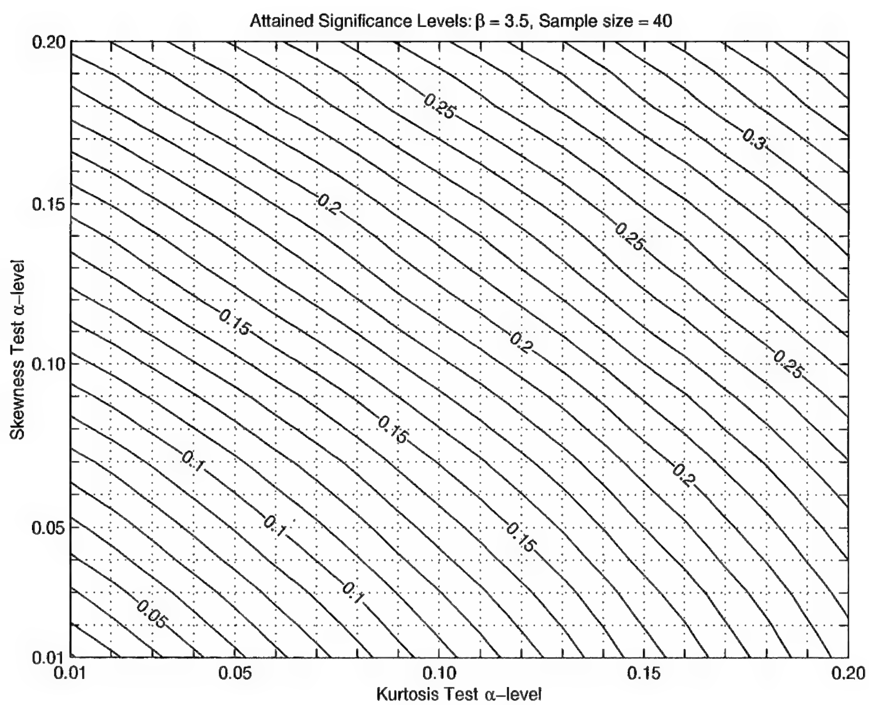
(e) Sample Size 25



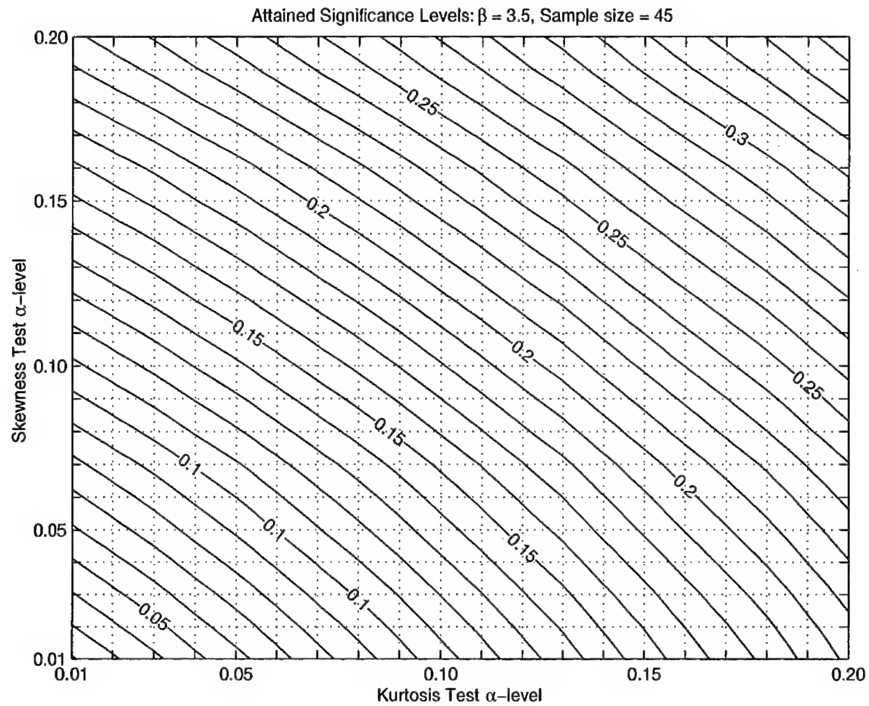
(f) Sample Size 30



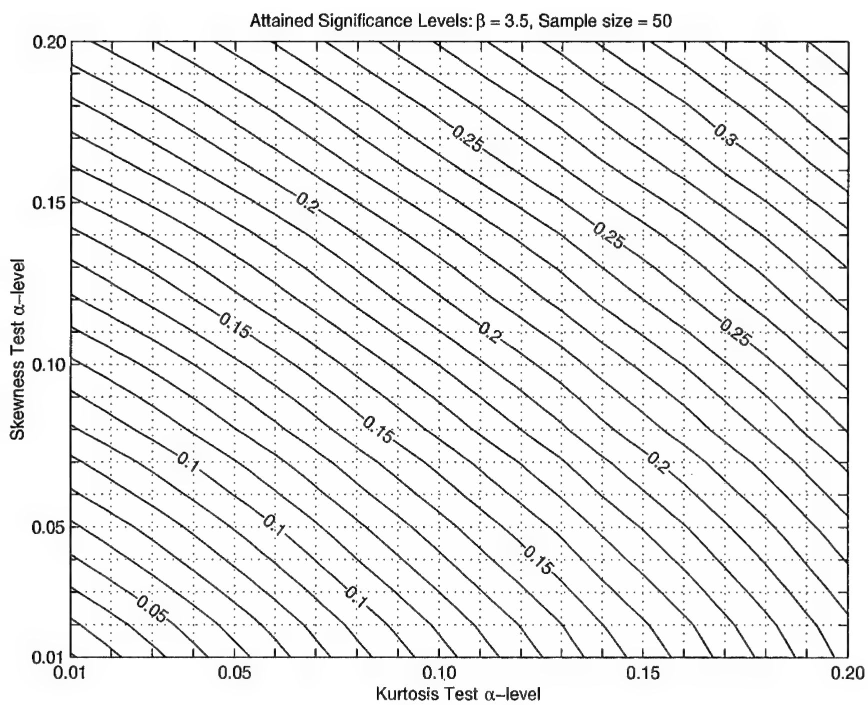
(g) Sample Size 35



(h) Sample Size 40

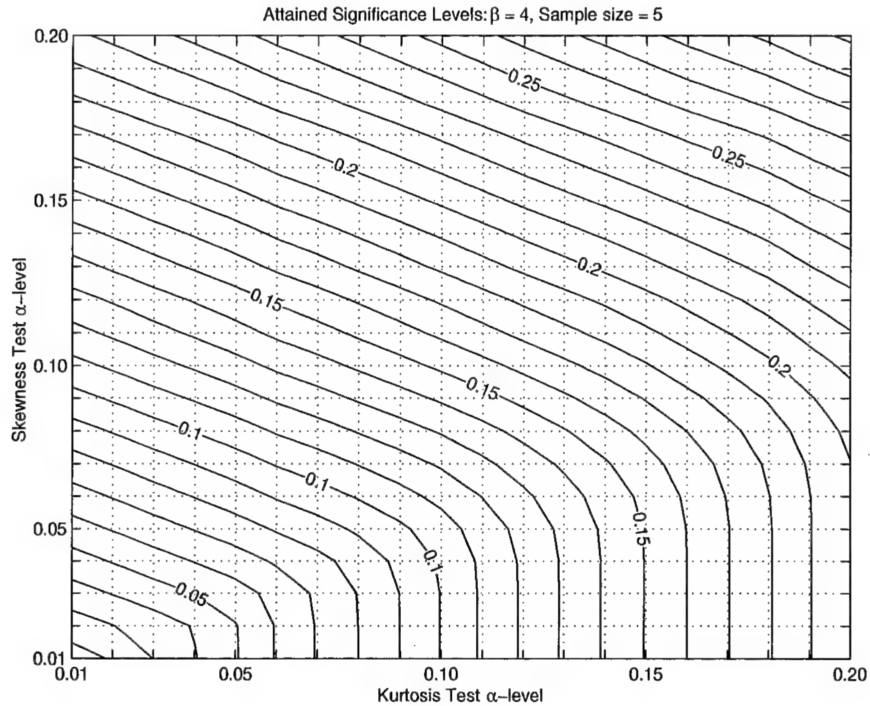


(i) Sample Size 45

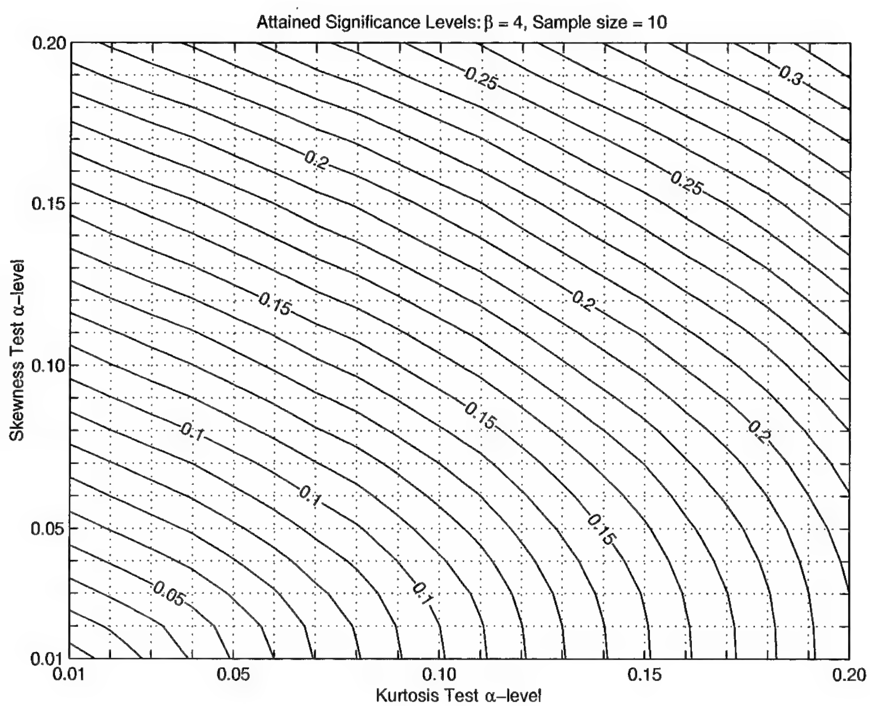


(j) Sample Size 50

C.8 Weibull Shape $\beta = 4$



(a) Sample Size 5



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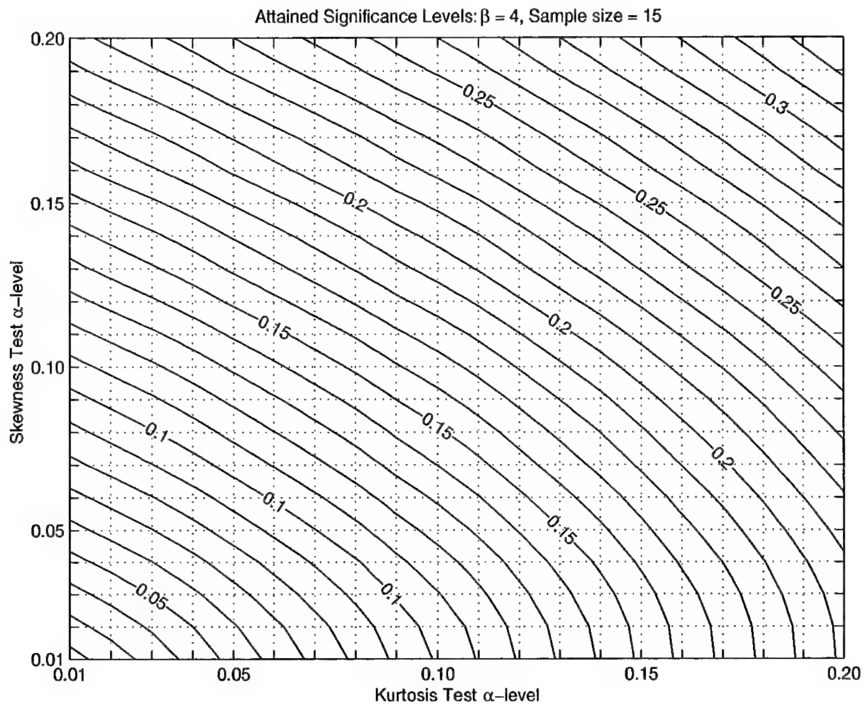
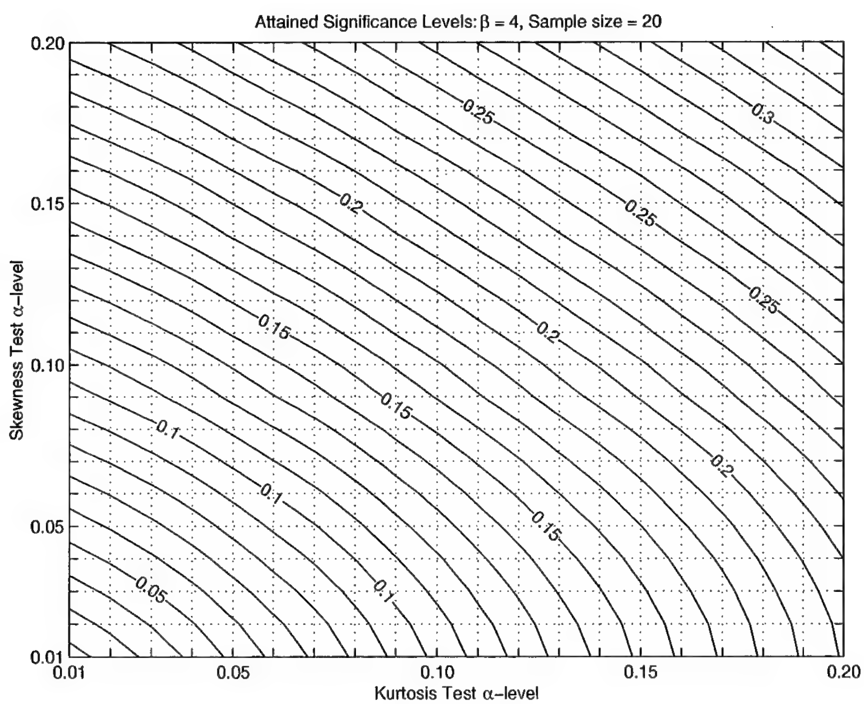
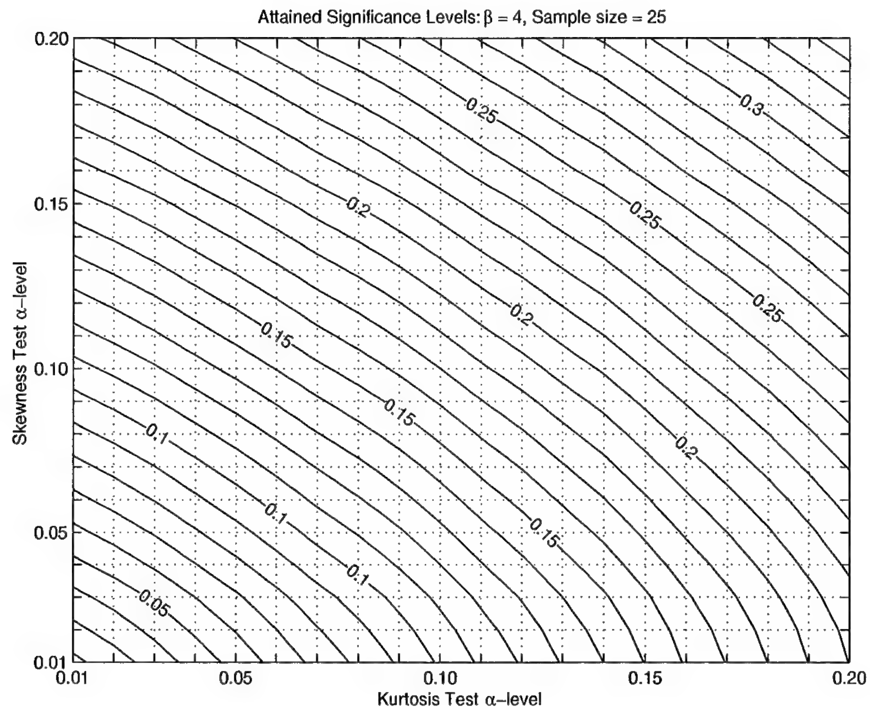


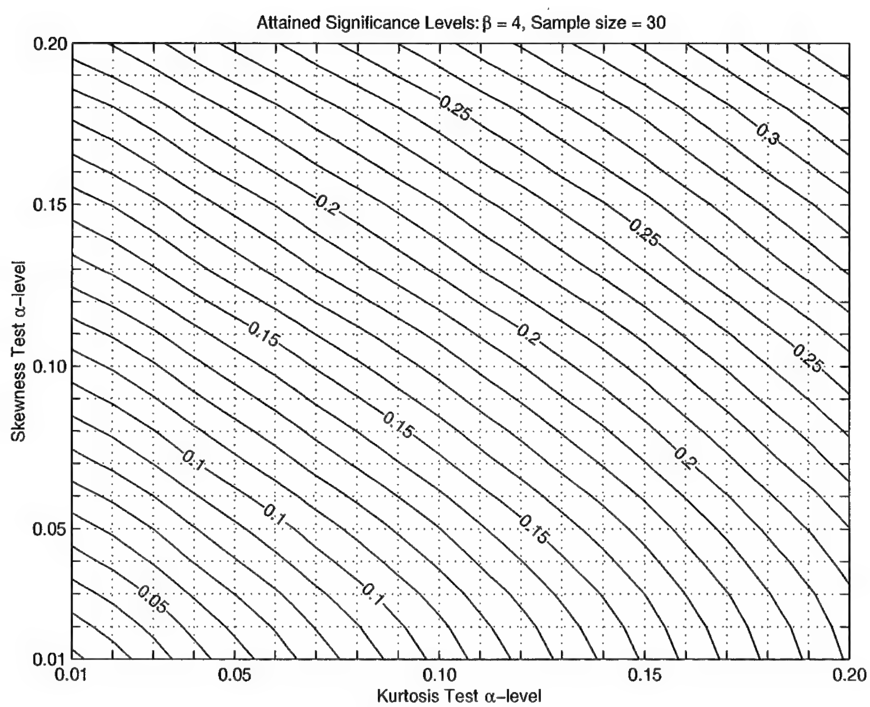
(c) Sample Size 15



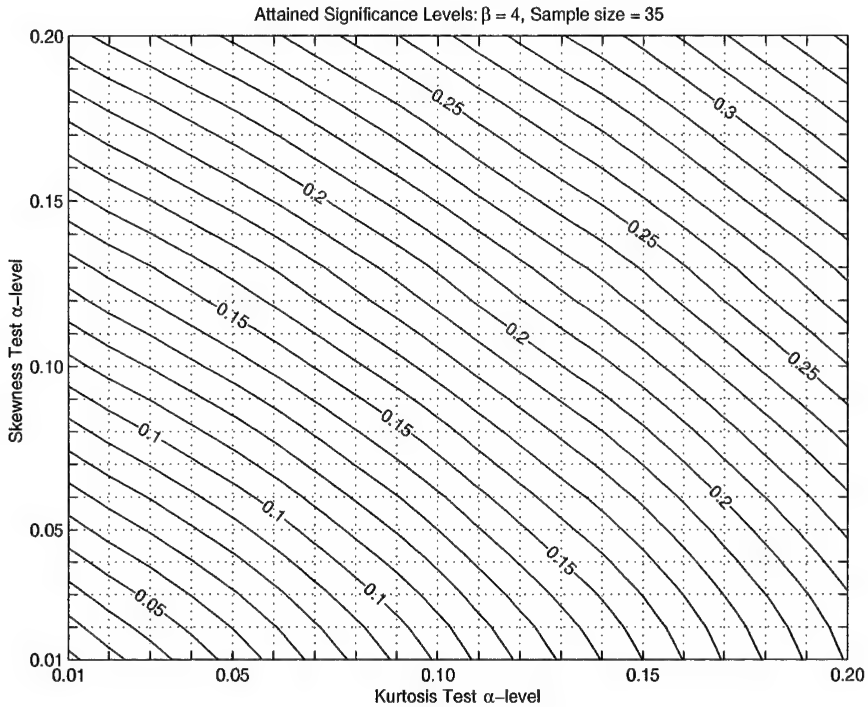
(d) Sample Size 20



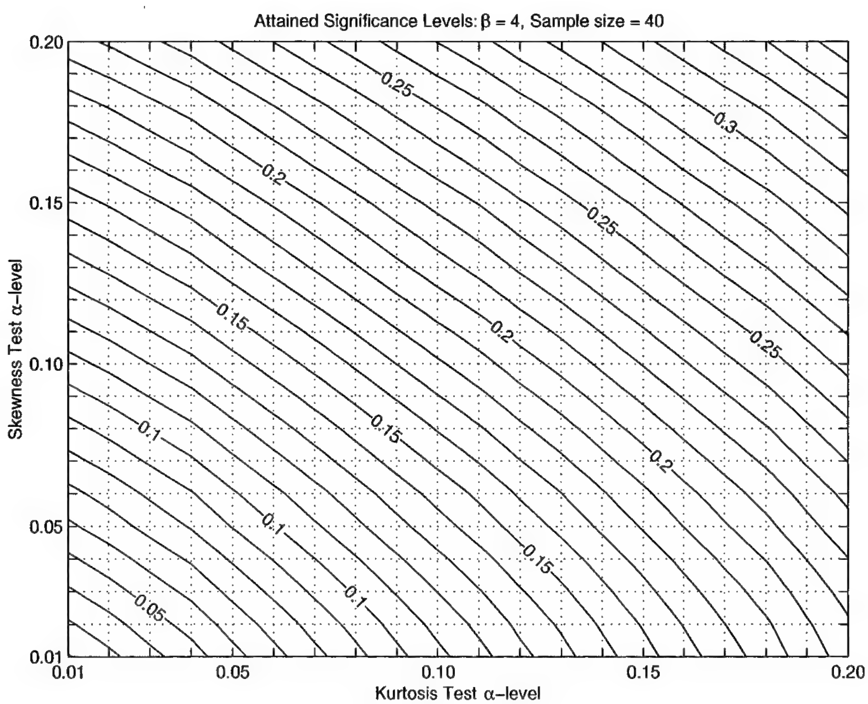
(e) Sample Size 25



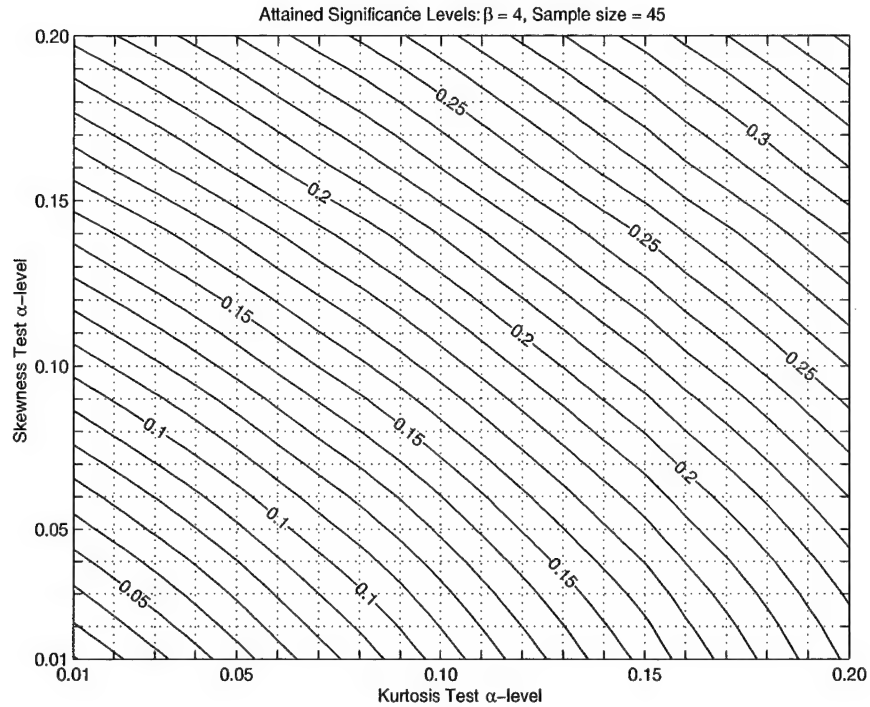
(f) Sample Size 30



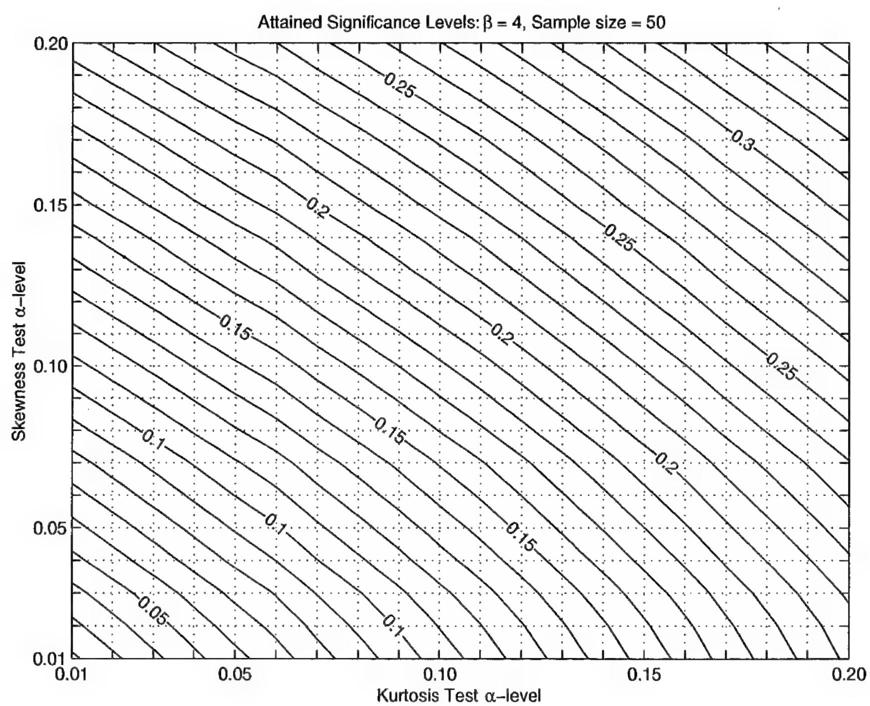
(g) Sample Size 35



(h) Sample Size 40



(i) Sample Size 45



(j) Sample Size 50

D.1 H_0: Weibull($\beta = 0.5$).

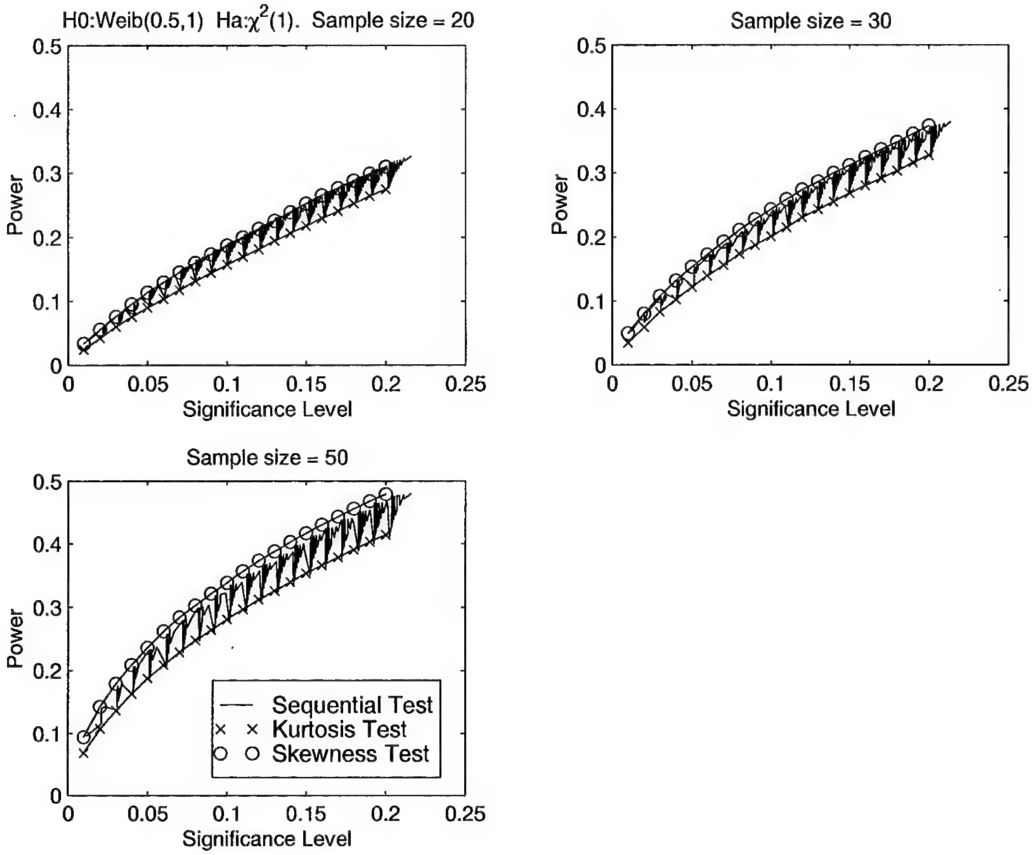


Figure D.1 Individual vs. Sequential Power: H_0:Weibull($\beta = 0.5$); H_a:$\chi^2(1)$.

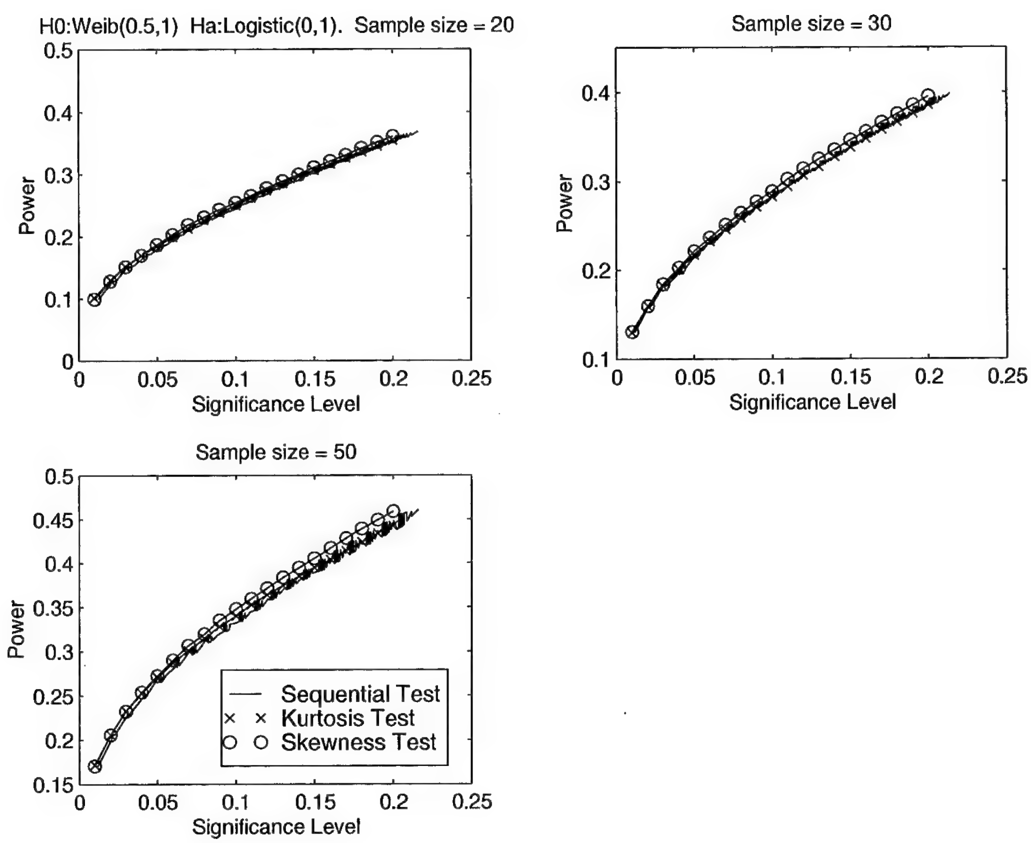


Figure D.2 Individual vs. Sequential Power: H_0:Weibull($\beta = 0.5$); H_a:XLogistic(0,1).

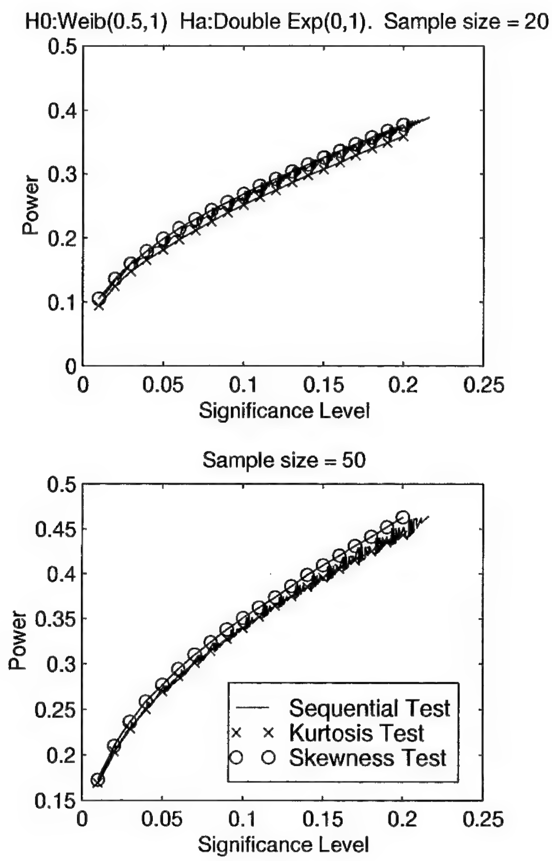


Figure D.3 Individual vs. Sequential Power: H_0:Weibull($\beta = 0.5$); H_a:XDouble Exponential.

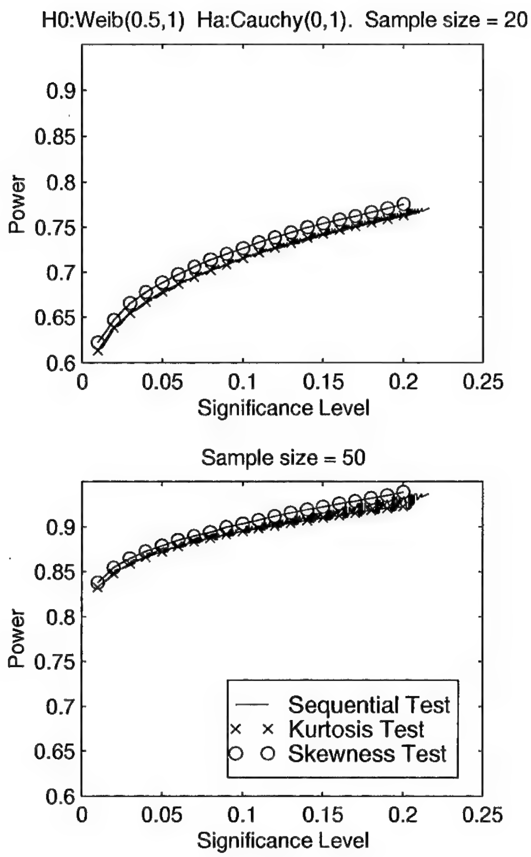


Figure D.4 Individual vs. Sequential Power: H_0:Weibull($\beta = 0.5$); H_a:XCauchy(0,1).

D.2 H_0: Weibull($\beta = 1$).

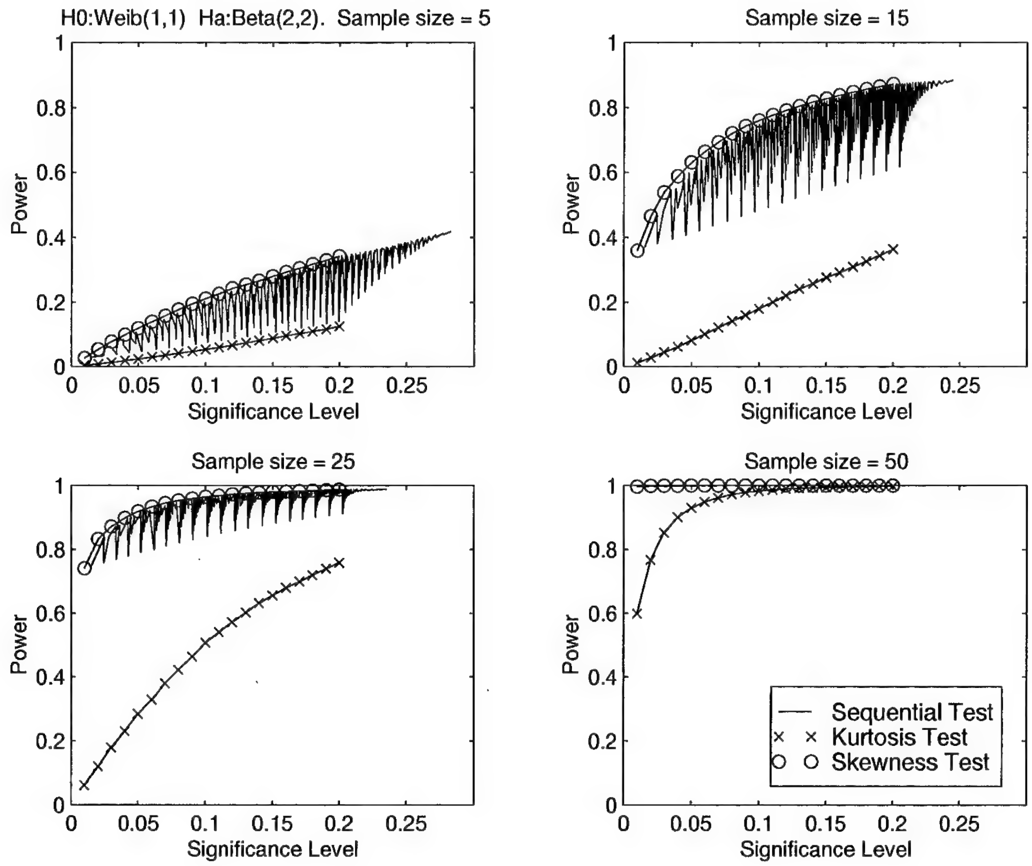


Figure D.5 Individual vs. Sequential Power: H_0:Weibull($\beta = 1$); H_a:Beta(2,2).

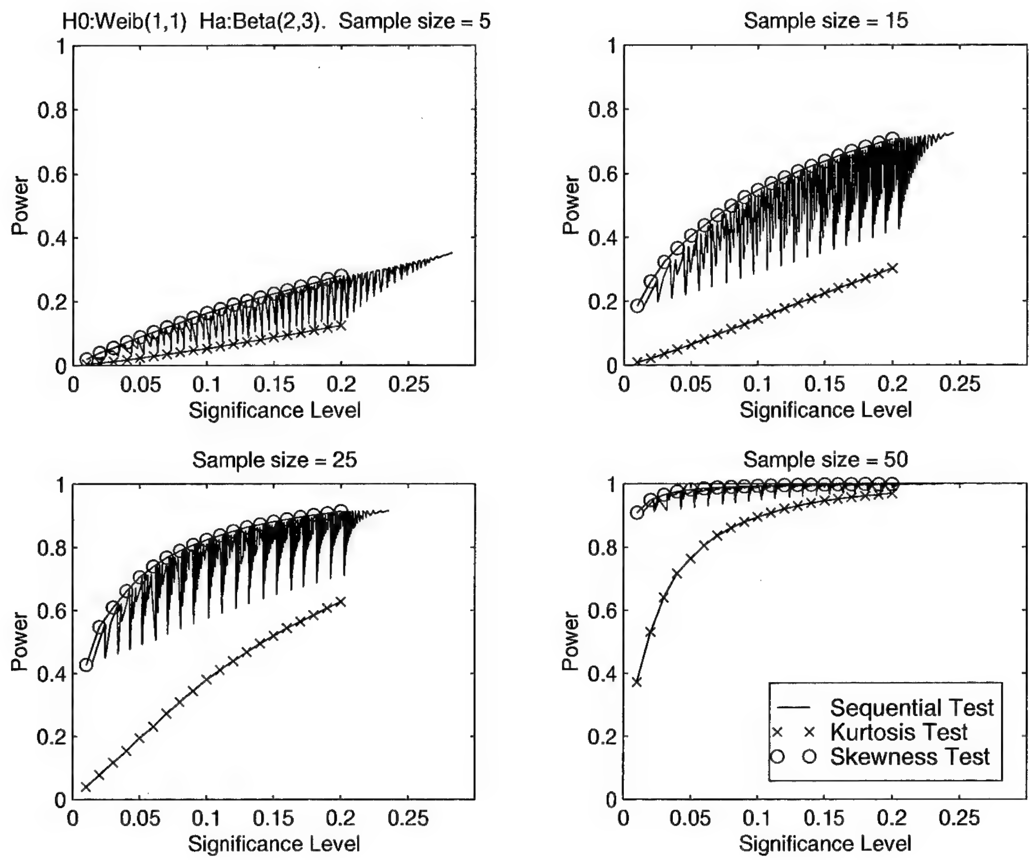


Figure D.6 Individual vs. Sequential Power: H_0:Weibull($\beta = 1$); H_a:Beta(2,3).

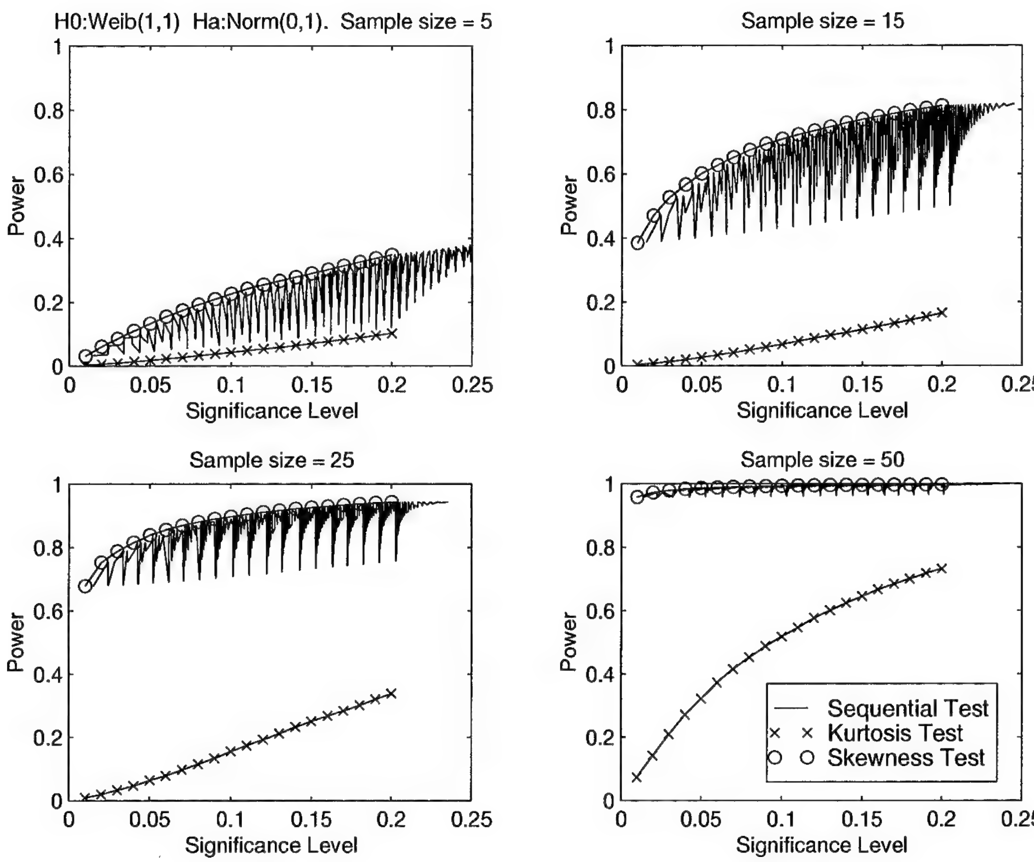


Figure D.7 Individual vs. Sequential Power: H_0:Weibull($\beta = 1$); H_a:Normal(0,1).

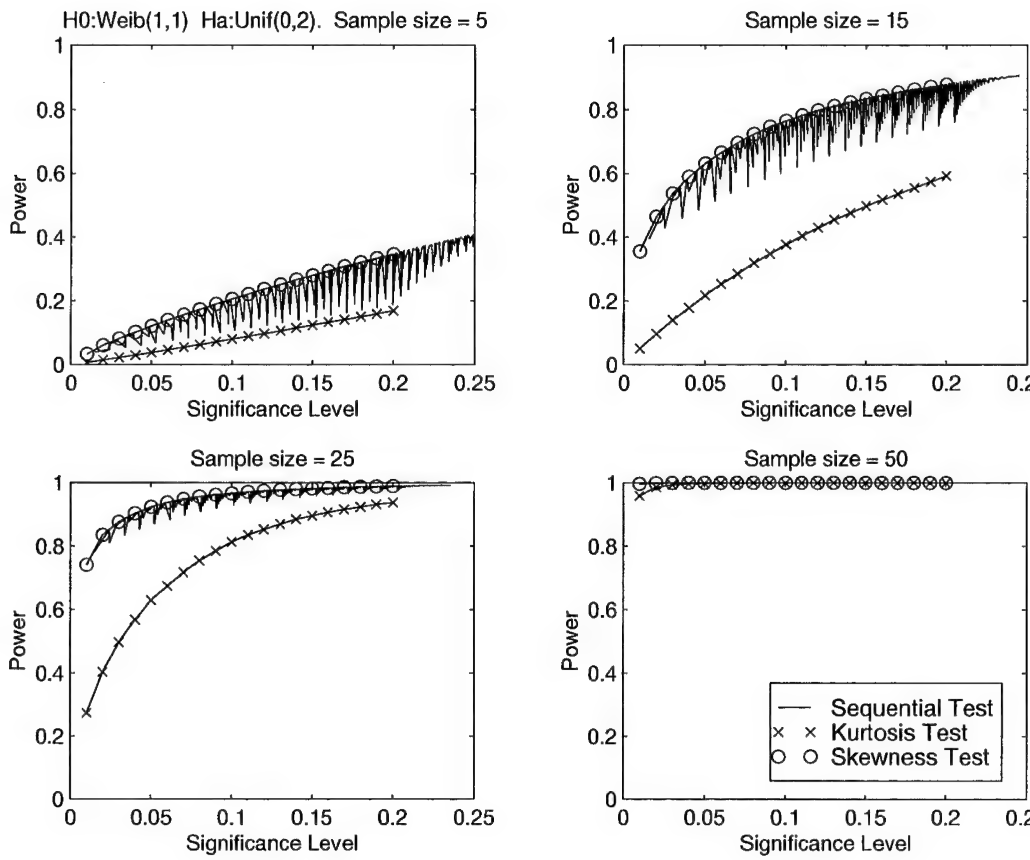


Figure D.8 Individual vs. Sequential Power: H_0:Weibull($\beta = 1$); H_a:Uniform(0,2).

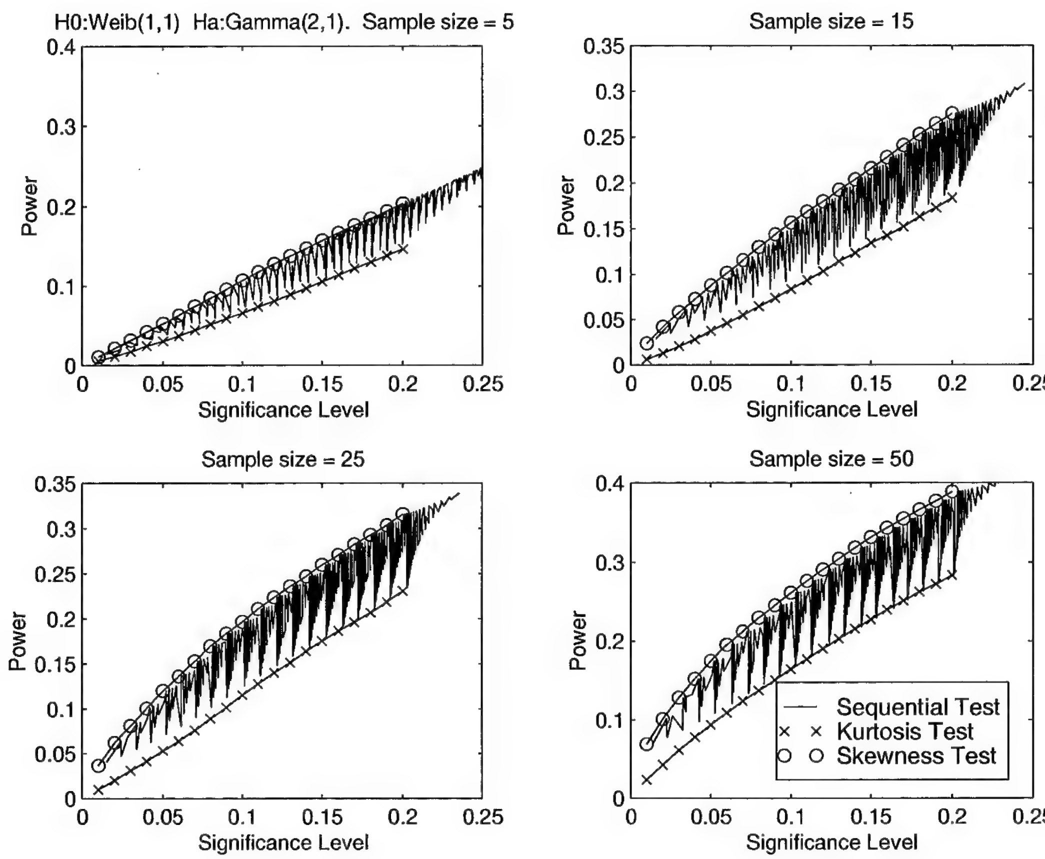


Figure D.9 Individual vs. Sequential Power: H_0:Weibull($\beta = 1$); H_a:Gamma(2,1).

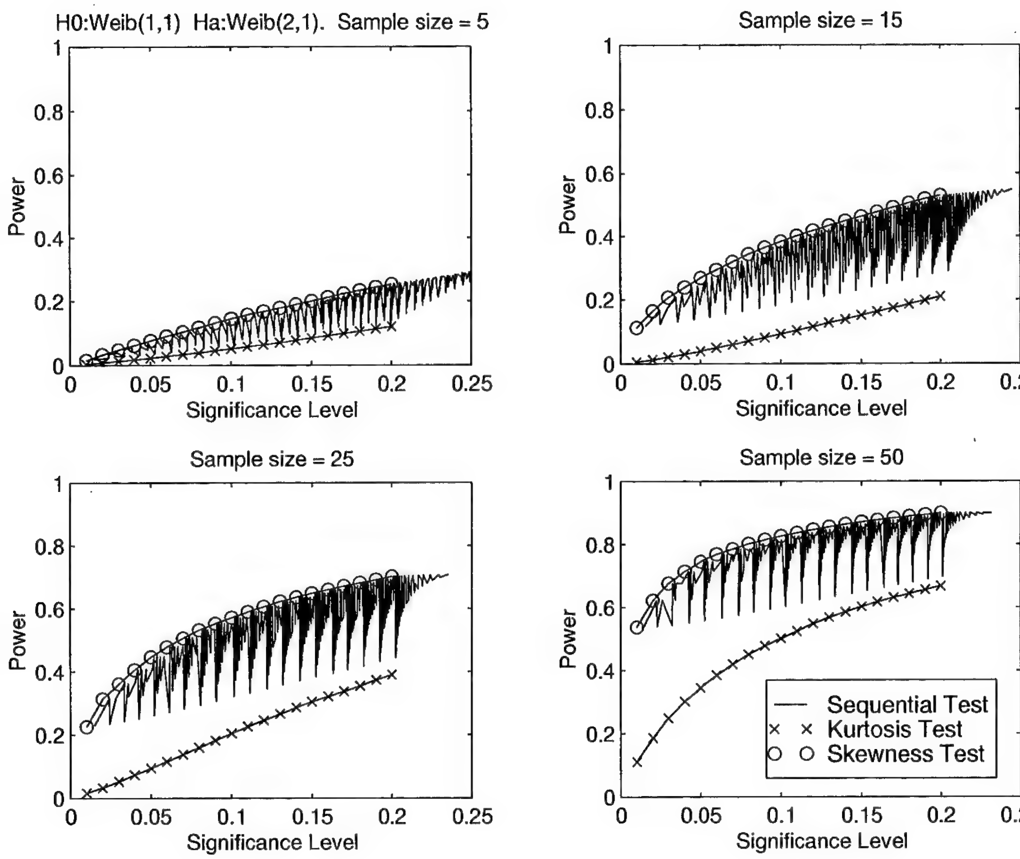


Figure D.10 Individual vs. Sequential Power: H_0:Weibull($\beta = 1$); H_a:Weibull($\beta = 2$).

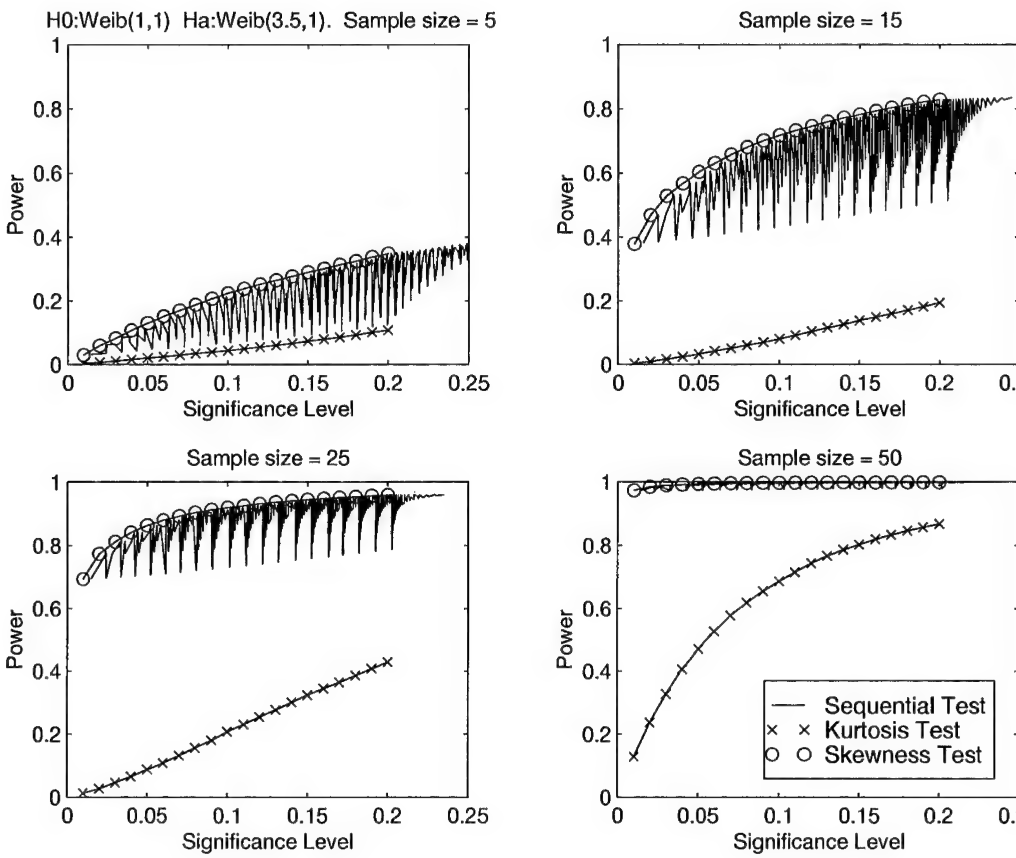


Figure D.11 Individual vs. Sequential Power: H_0:Weibull($\beta = 1$); H_a:Weibull($\beta = 3.5$).

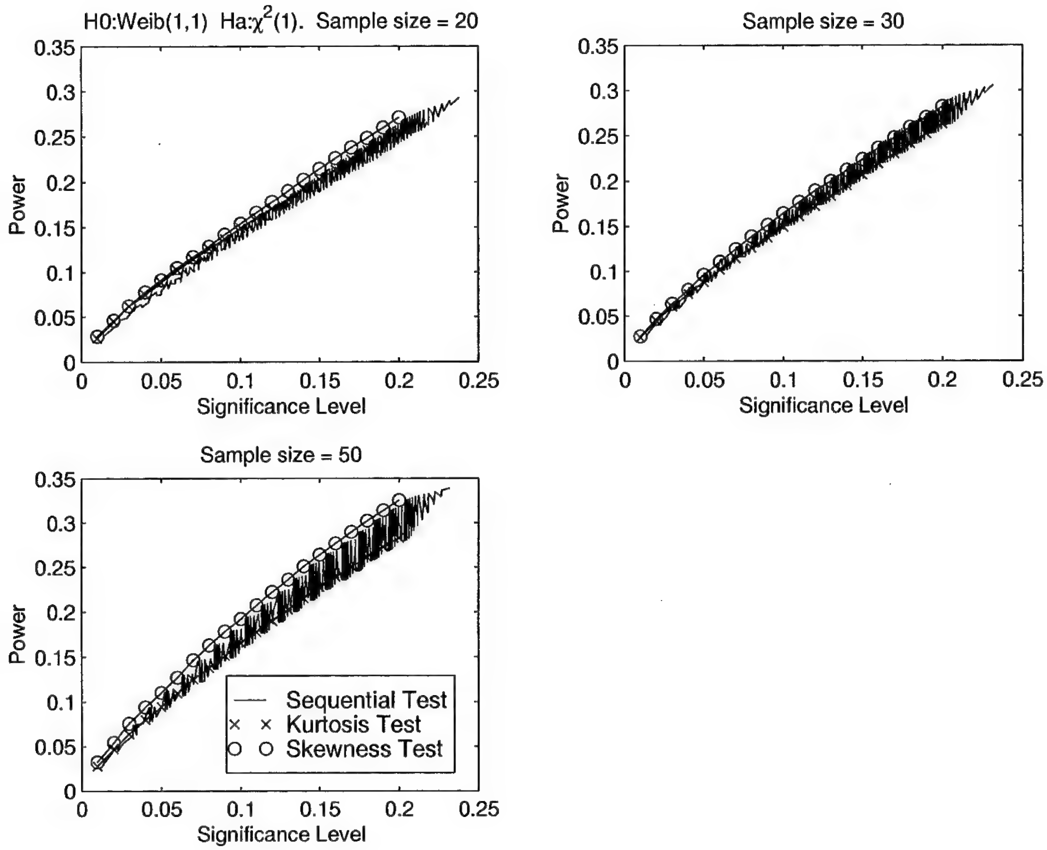


Figure D.12 Individual vs. Sequential Power: H_0:Weibull($\beta = 1$); H_a:$\chi^2(1)$.

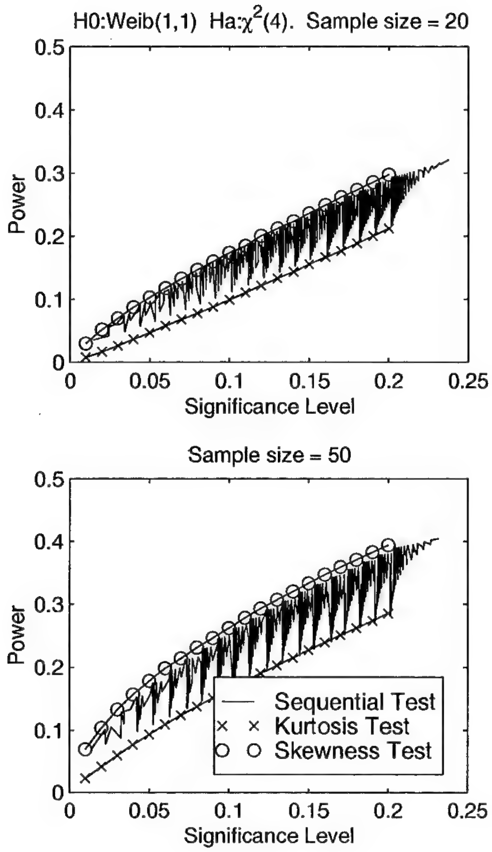


Figure D.13 Individual vs. Sequential Power: H_0:Weibull($\beta = 1$); H_a:$\chi^2(4)$.

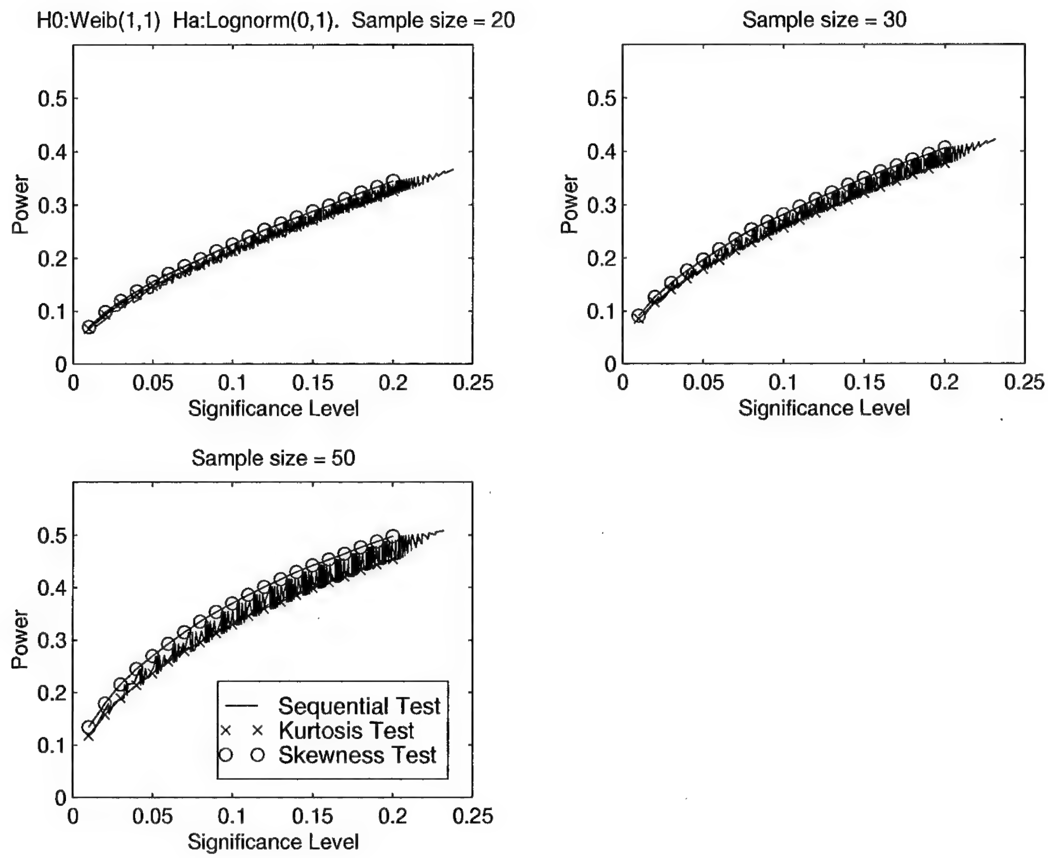


Figure D.14 Individual vs. Sequential Power: H_0:Weibull($\beta = 1$); H_a:Lognormal(0,1).

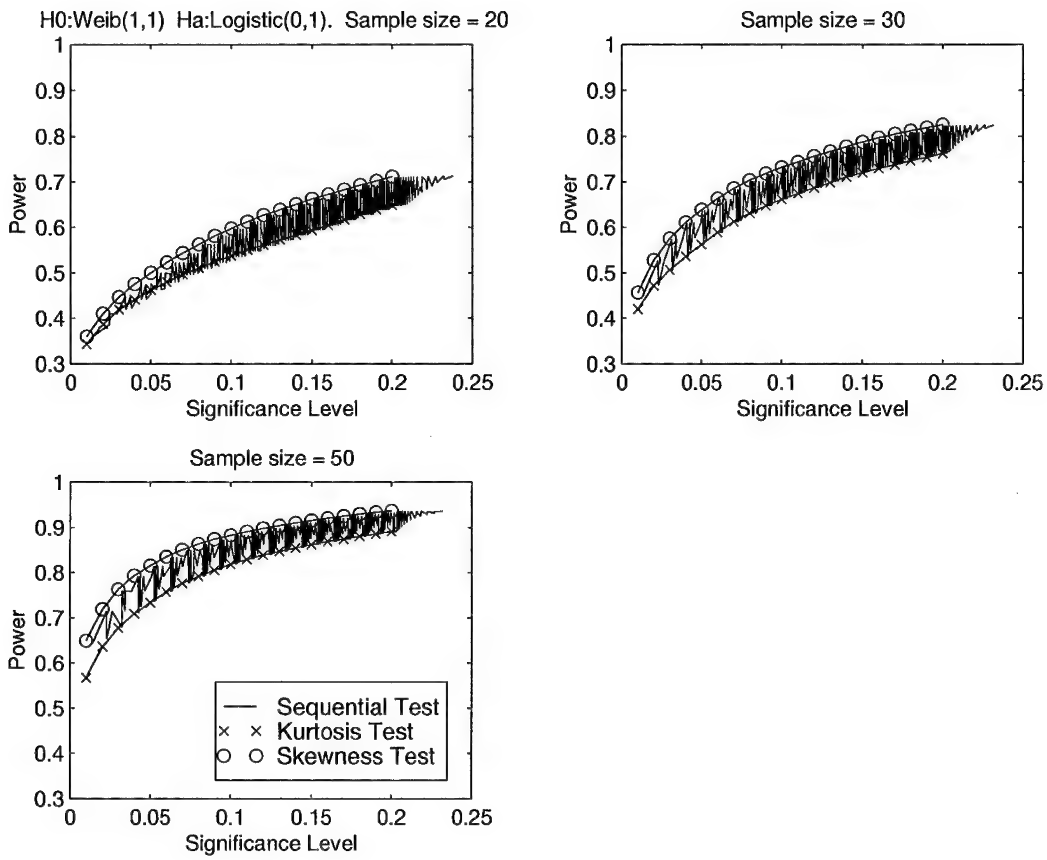


Figure D.15 Individual vs. Sequential Power: H_0:Weibull($\beta = 1$); H_a:XLogistic(0,1).

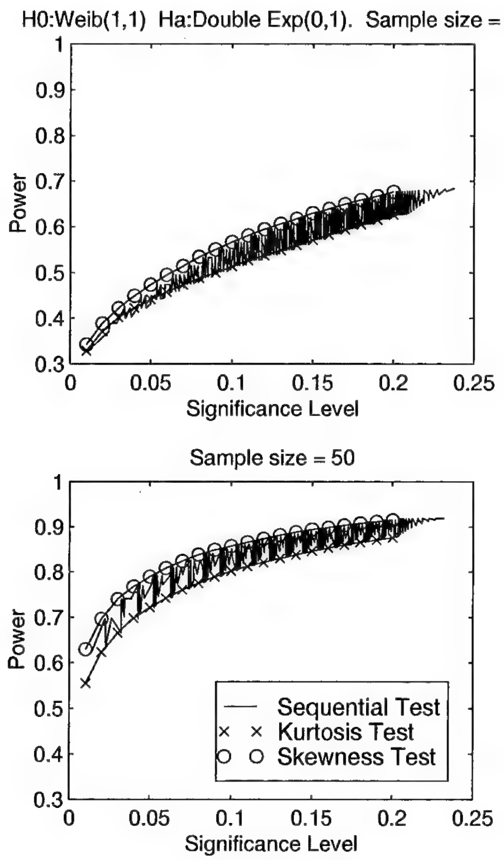


Figure D.16 Individual vs. Sequential Power: H_0:Weibull($\beta = 1$); H_a:XDouble Exponential.

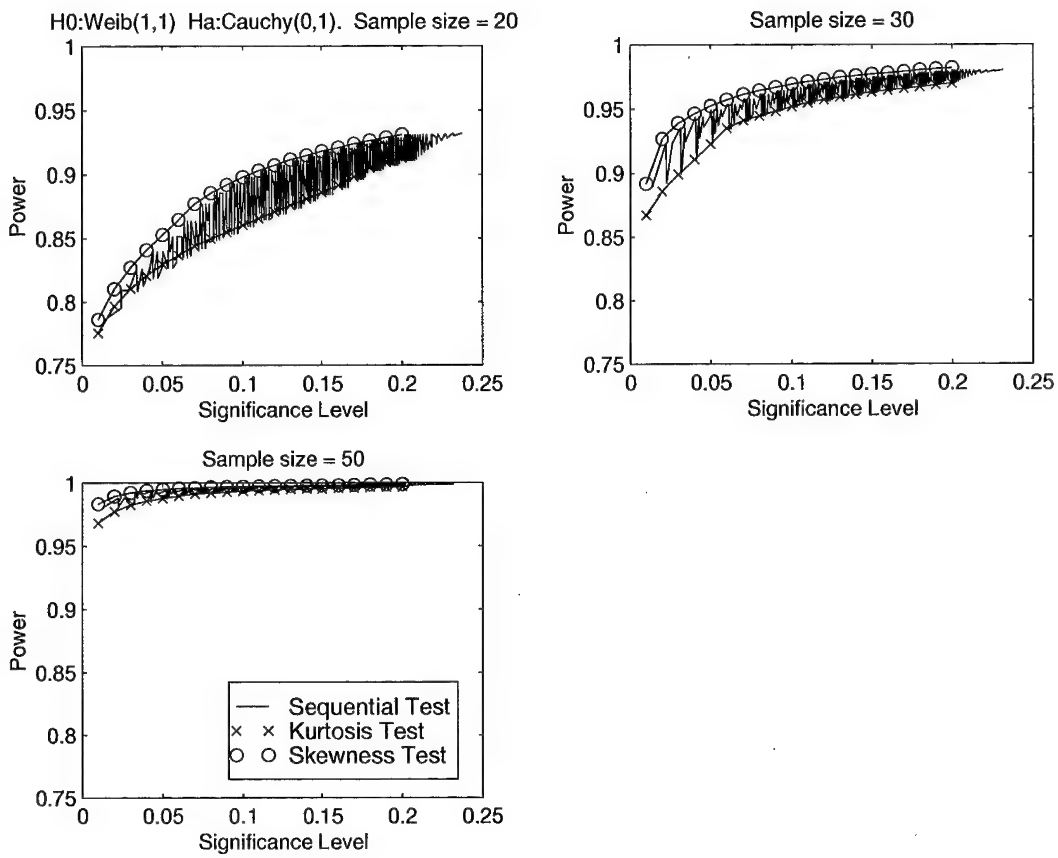


Figure D.17 Individual vs. Sequential Power: H_0:Weibull($\beta = 1$); H_a:XCauchy(0,1).

D.3 H_0: Weibull($\beta = 1.5$).

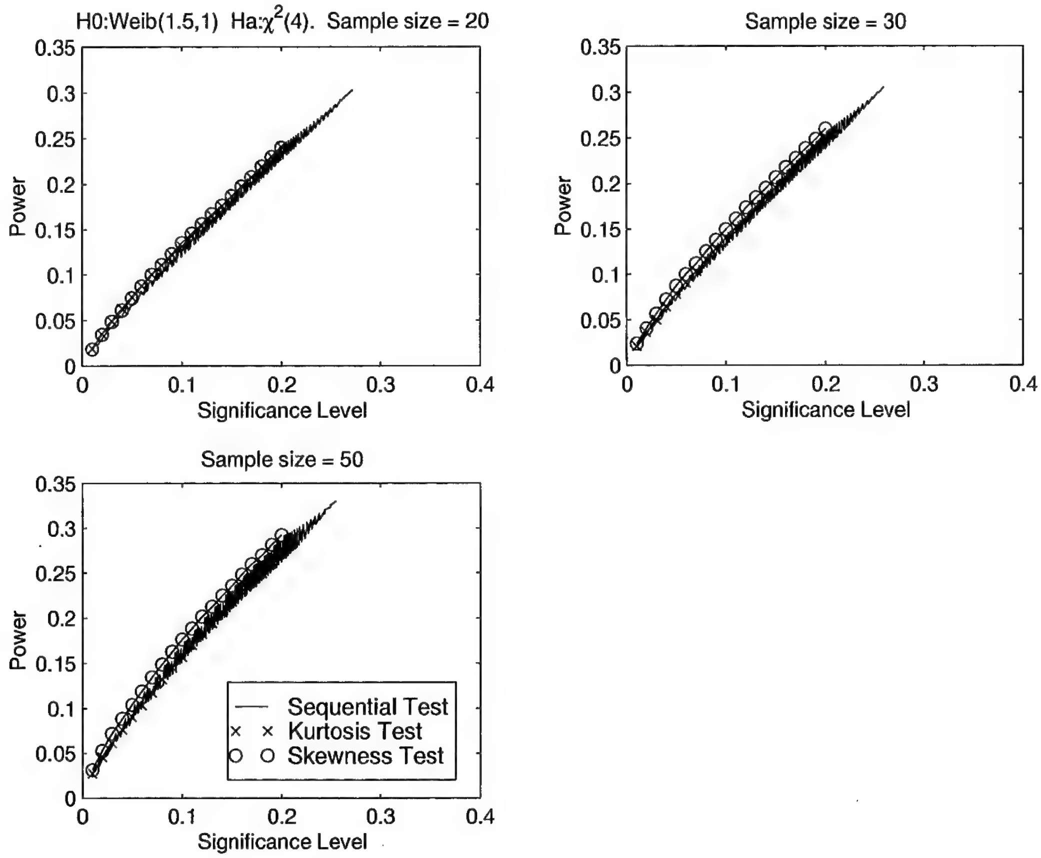


Figure D.18 Individual vs. Sequential Power: H_0:Weibull($\beta = 1.5$); H_a:$\chi^2(4)$.

D.4 H_0: *Weibull*($\beta = 3.5$).

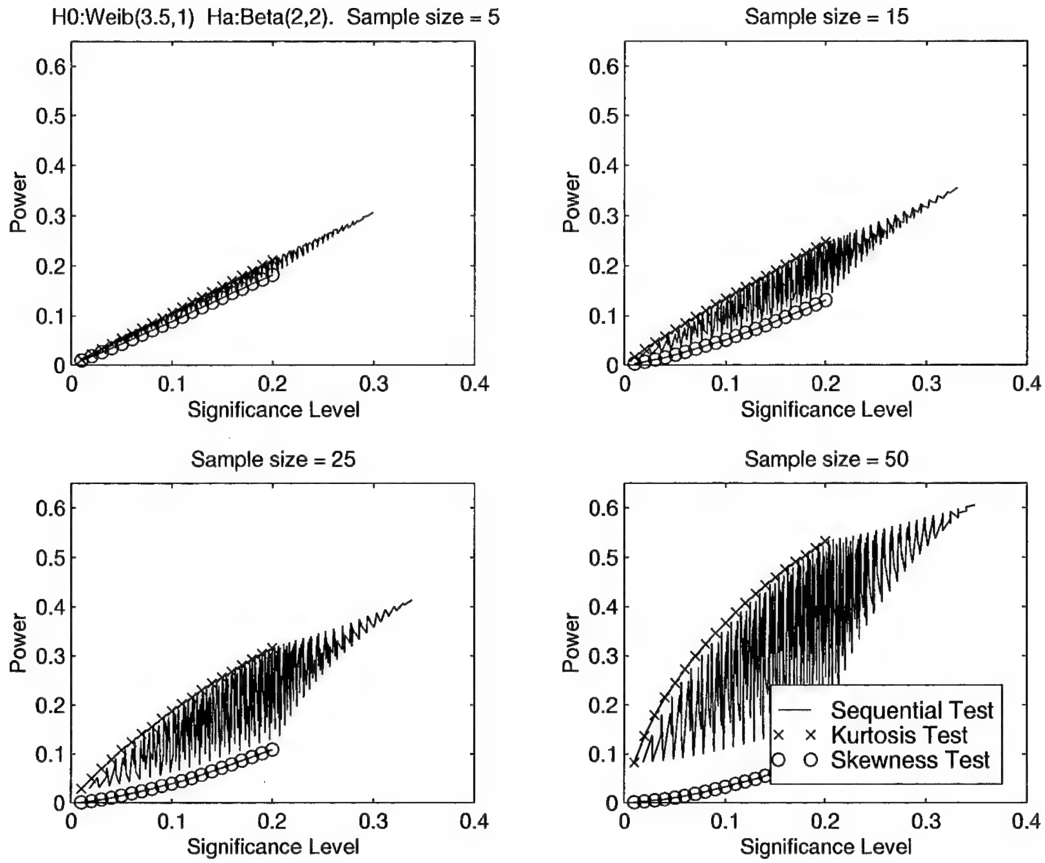


Figure D.19 Individual vs. Sequential Power: H_0:Weibull($\beta = 3.5$); H_a:Beta(2,2).

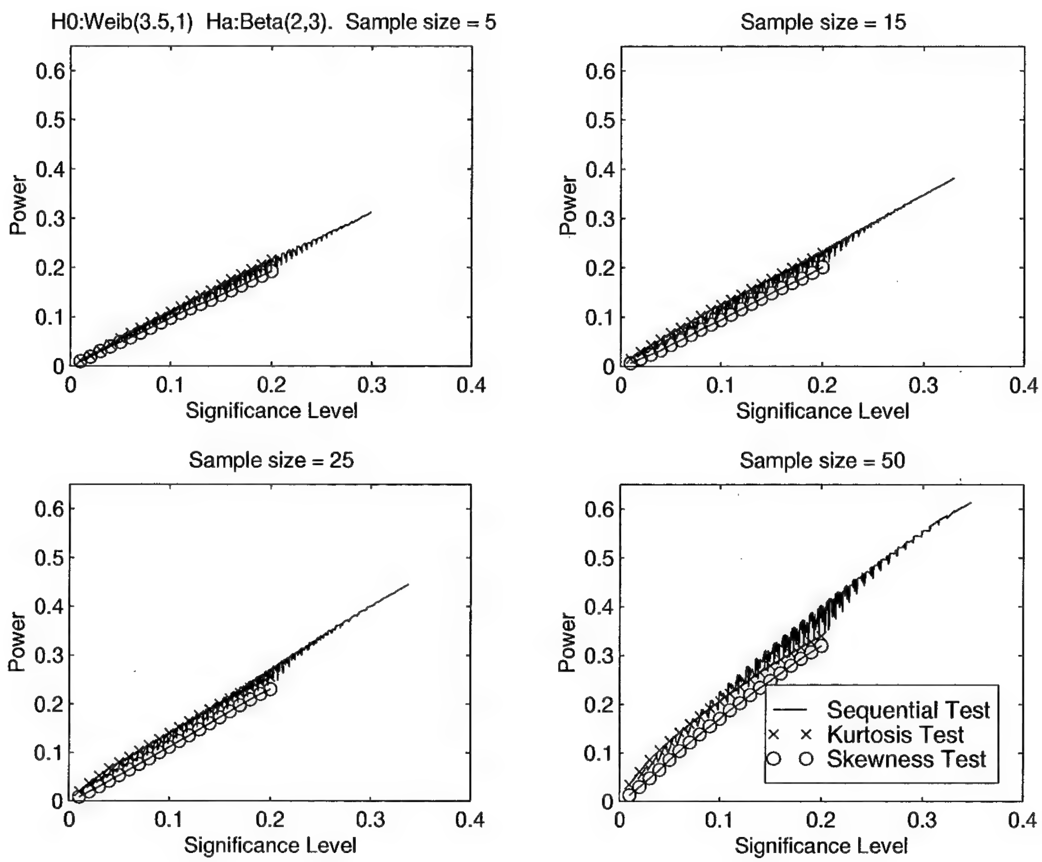


Figure D.20 Individual vs. Sequential Power: H_0:Weibull($\beta = 3.5$); H_a:Beta(2,3).

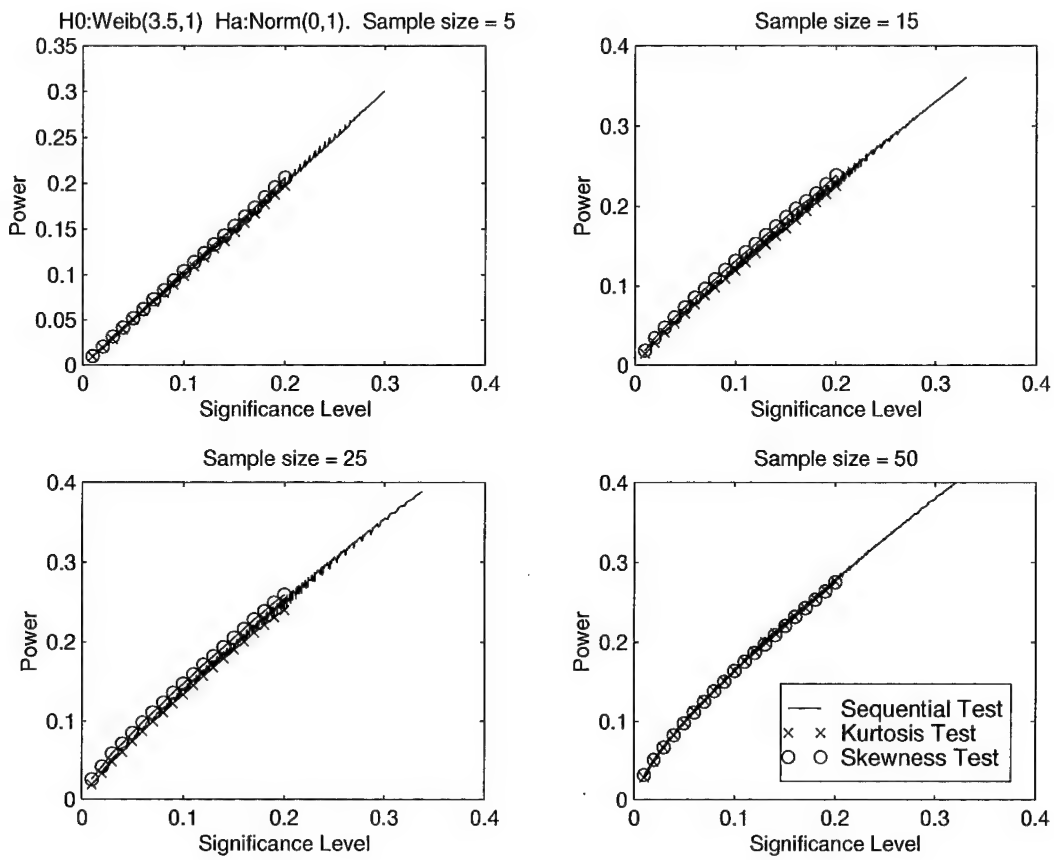


Figure D.21 Individual vs. Sequential Power: H_0:Weibull($\beta = 3.5$); H_a:Normal(0,1).

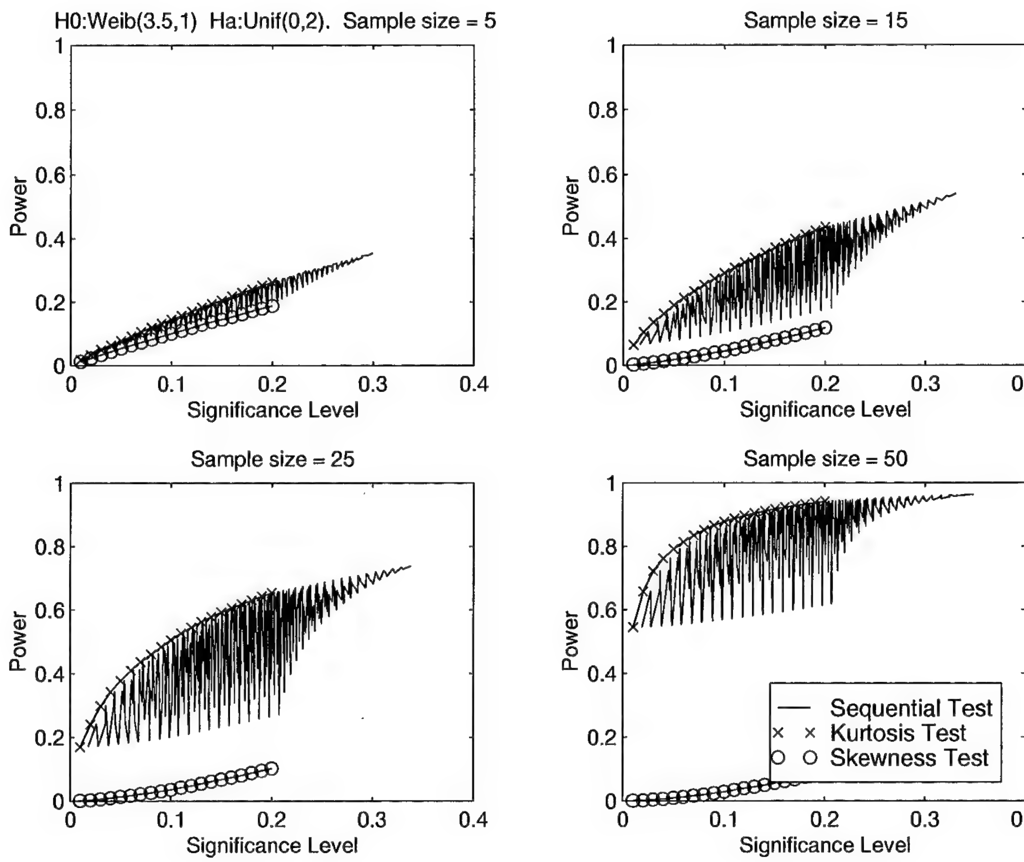


Figure D.22 Individual vs. Sequential Power: H_0:Weibull($\beta = 3.5$); H_a:Uniform(0,2).

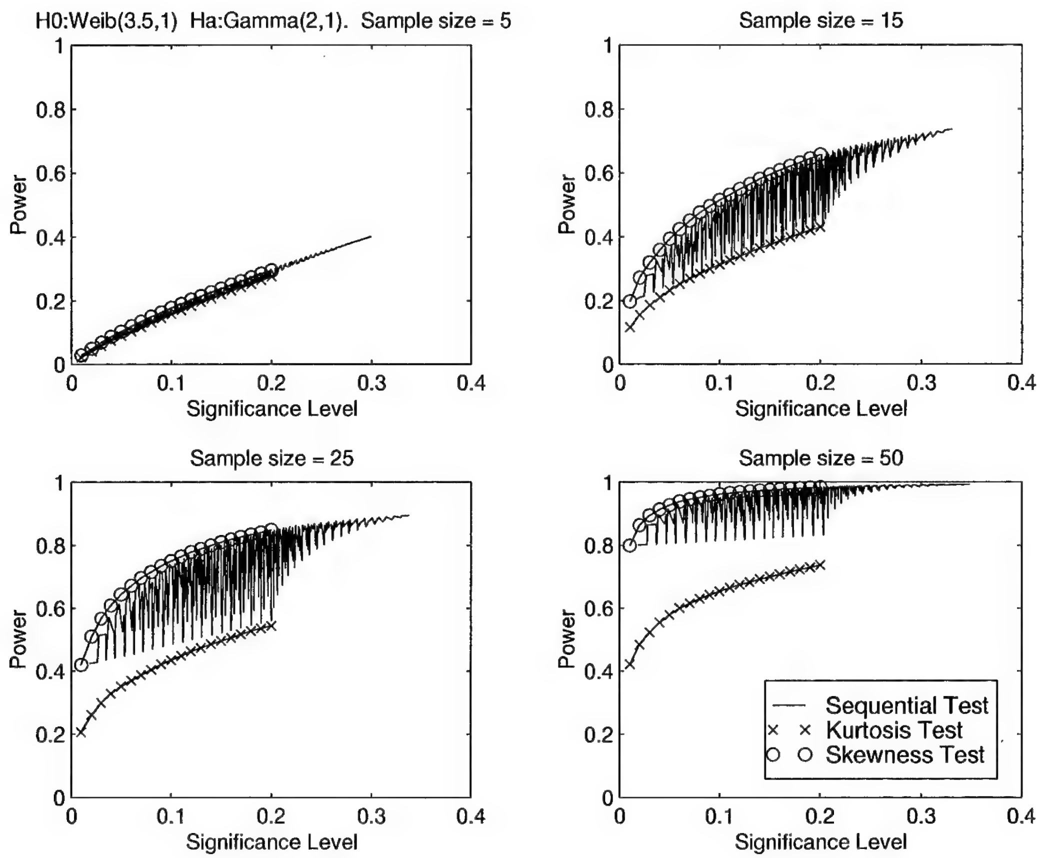


Figure D.23 Individual vs. Sequential Power: H_0:Weibull($\beta = 3.5$); H_a:Gamma(2,1).

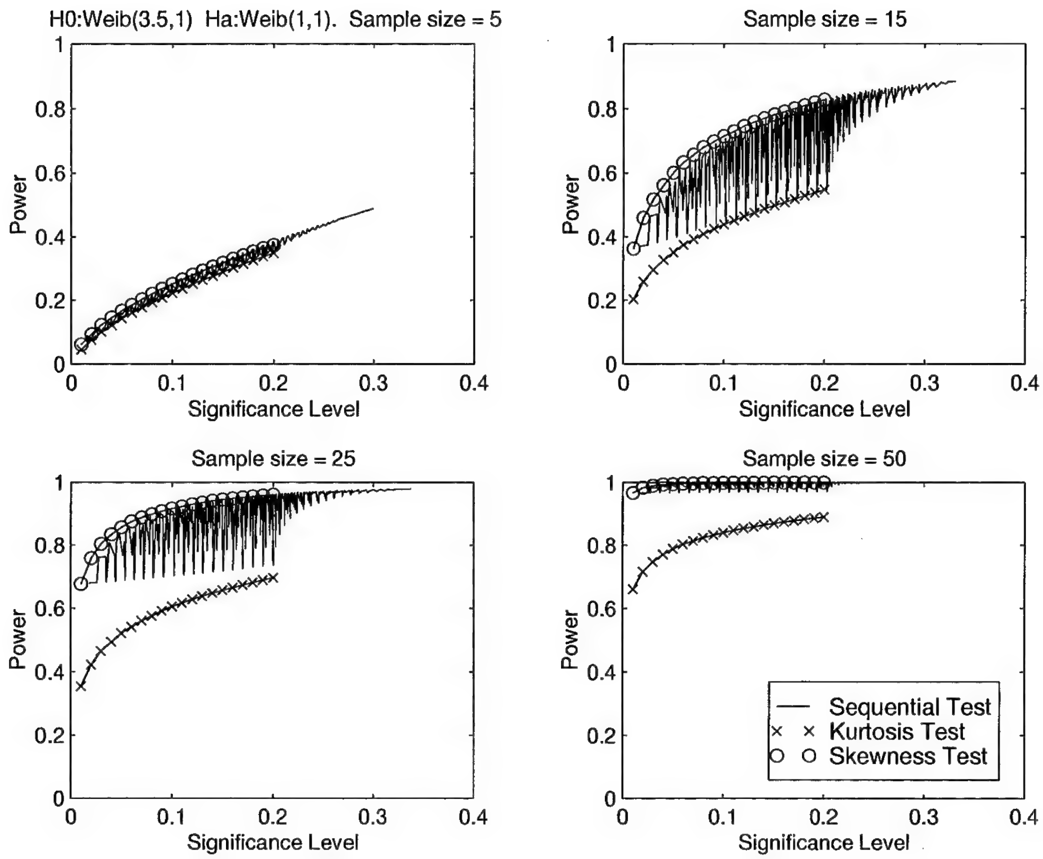


Figure D.24 Individual vs. Sequential Power: H_0:Weibull($\beta = 3.5$); H_a:Weibull($\beta = 1$).

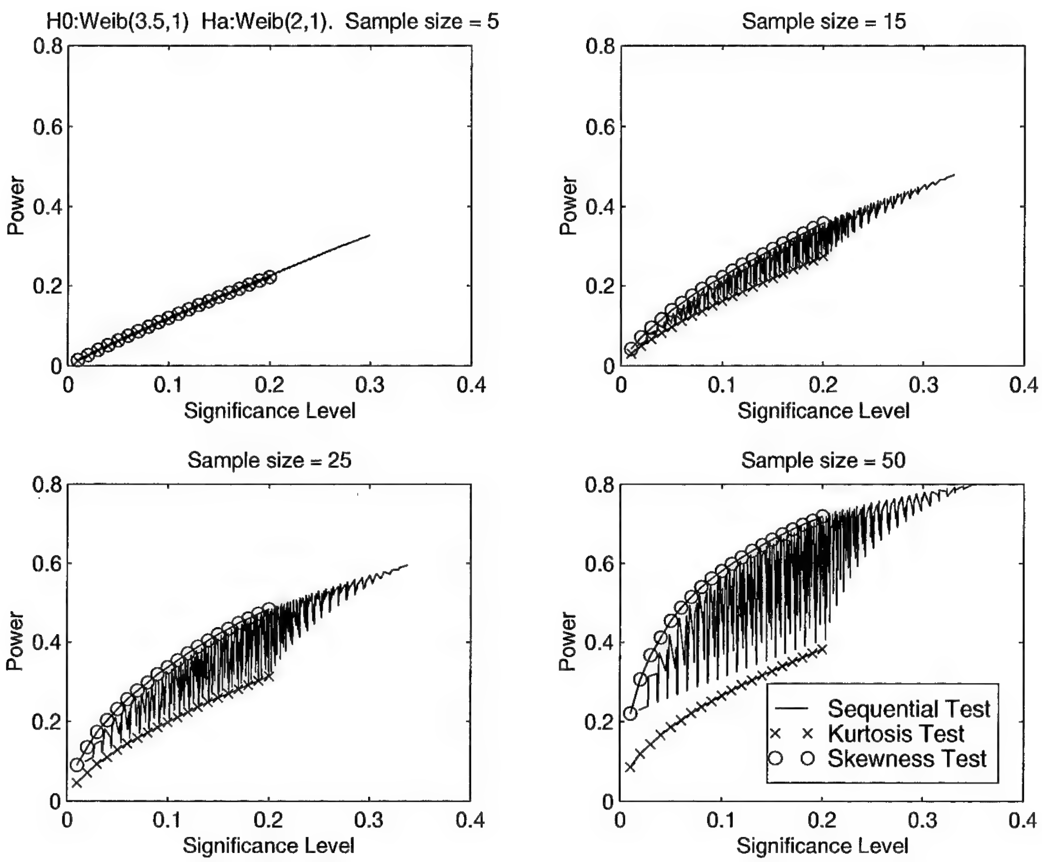


Figure D.25 Individual vs. Sequential Power: H_0:Weibull($\beta = 3.5$); H_a:Weibull($\beta = 2$).

Appendix E. Individual Skewness and Kurtosis Test Power Results (Two-sided)

E.1 H_0: Weibull($\beta = 0.5$)

E.1.1 Skewness Test Results.

Table E.1 Power Study: Skewness Test – H_0: Weibull(0.5) ; H_a: $\chi^2(1)$.

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
20	0.034	0.114	0.188	0.253	0.310
30	0.050	0.154	0.243	0.311	0.374
50	0.094	0.236	0.339	0.417	0.479

Table E.2 Power Study: Skewness Test – H_0: Weibull(0.5) ; H_a: XDouble Exp.

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
20	0.105	0.199	0.269	0.326	0.377
30	0.132	0.228	0.298	0.356	0.405
50	0.173	0.277	0.351	0.409	0.463

Table E.3 Power Study: Skewness Test – H_0: Weibull(0.5) ; H_a: XLogistic(0,1).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
20	0.099	0.187	0.254	0.311	0.361
30	0.131	0.221	0.289	0.346	0.395
50	0.171	0.272	0.348	0.405	0.459

Table E.4 Power Study: Skewness Test – H_0: Weibull(0.5) ; H_a: XCauchy(0,1).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
20	0.622	0.688	0.726	0.754	0.776
30	0.730	0.784	0.814	0.836	0.853
50	0.838	0.879	0.903	0.922	0.938

E.1.2 Kurtosis Test Results.

Table E.5 Power Study: Kurtosis Test – H_0: Weibull(0.5) ; H_a: $\chi^2(1)$.

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
20	0.025	0.090	0.158	0.218	0.275
30	0.034	0.122	0.201	0.268	0.328
50	0.068	0.188	0.281	0.354	0.414

Table E.6 Power Study: Kurtosis Test – H_0: Weibull(0.5) ; H_a: XDouble Exp.

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
20	0.094	0.182	0.251	0.307	0.359
30	0.129	0.223	0.293	0.349	0.398
50	0.169	0.270	0.339	0.396	0.443

Table E.7 Power Study: Kurtosis Test – H_0: Weibull(0.5) ; H_a: XLogistic(0,1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
20	0.102	0.185	0.250	0.307	0.355
30	0.130	0.217	0.283	0.338	0.386
50	0.172	0.271	0.340	0.395	0.442

Table E.8 Power Study: Kurtosis Test – H_0: Weibull(0.5) ; H_a: XCauchy(0,1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
20	0.614	0.678	0.715	0.743	0.763
30	0.725	0.776	0.806	0.827	0.844
50	0.833	0.872	0.895	0.910	0.922

E.2 H_0: Weibull($\beta = 1$)

E.2.1 Skewness Test Results.

Table E.9 Power Study: Skewness Test – H_0: Weibull(1) ; H_a: Beta(2,2).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.026	0.119	0.211	0.280	0.342
15	0.360	0.631	0.760	0.827	0.872
25	0.741	0.919	0.964	0.980	0.988
50	0.996	1.000	1.000	1.000	1.000

Table E.10 Power Study: Skewness Test – H_0: Weibull(1) ; H_a: Beta(2,3).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.019	0.089	0.163	0.224	0.280
15	0.185	0.406	0.549	0.638	0.707
25	0.427	0.706	0.824	0.881	0.914
50	0.909	0.981	0.993	0.997	0.998

Table E.11 Power Study: Skewness Test – H_0: Weibull(1) ; H_a: Gamma(2,1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.011	0.053	0.107	0.157	0.203
15	0.024	0.088	0.157	0.216	0.276
25	0.037	0.120	0.197	0.260	0.316
50	0.069	0.175	0.261	0.332	0.389

Table E.12 Power Study: Skewness Test – H_0: Weibull(1) ; H_a: Normal(0,1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.031	0.133	0.226	0.291	0.348
15	0.385	0.601	0.709	0.770	0.812
25	0.678	0.839	0.898	0.926	0.943
50	0.959	0.986	0.993	0.996	0.997

Table E.13 Power Study: Skewness Test – H_0: Weibull(1) ; H_a: Unif(0,2).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.033	0.121	0.206	0.280	0.345
15	0.355	0.632	0.765	0.834	0.878
25	0.741	0.924	0.966	0.982	0.989
50	0.997	1.000	1.000	1.000	1.000

Table E.14 Power Study: Skewness Test – H_0: Weibull(1) ; H_a: Weibull(1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.010	0.050	0.101	0.151	0.200
15	0.010	0.049	0.098	0.148	0.201
25	0.010	0.050	0.102	0.153	0.200
50	0.010	0.050	0.101	0.150	0.199

Table E.15 Power Study: Skewness Test – H_0: Weibull(1) ; H_a: Weibull(2,1,0).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.015	0.076	0.145	0.200	0.253
15	0.112	0.268	0.383	0.463	0.530
25	0.226	0.447	0.572	0.650	0.703
50	0.537	0.745	0.826	0.872	0.899

Table E.16 Power Study: Skewness Test – H_0: Weibull(1) ; H_a: Weibull(3.5,1,0).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.029	0.130	0.224	0.290	0.348
15	0.378	0.604	0.717	0.781	0.827
25	0.694	0.864	0.919	0.944	0.959
50	0.974	0.994	0.997	0.998	0.999

Table E.17 Power Study: Skewness Test – H_0: Weibull(1) ; H_a: $\chi^2(1)$.

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
20	0.029	0.091	0.154	0.215	0.271
30	0.027	0.096	0.164	0.224	0.282
50	0.033	0.110	0.192	0.264	0.326

Table E.18 Power Study: Skewness Test – H_0: Weibull(1) ; H_a: $\chi^2(4)$.

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
20	0.030	0.103	0.174	0.236	0.298
30	0.042	0.130	0.210	0.277	0.334
50	0.070	0.178	0.262	0.333	0.394

Table E.19 Power Study: Skewness Test – H_0: Weibull(1) ; H_a: Lognorm(0,1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
20	0.070	0.155	0.226	0.288	0.345
30	0.091	0.197	0.282	0.350	0.406
50	0.134	0.270	0.370	0.442	0.498

Table E.20 Power Study: Skewness Test – H_0: Weibull(1) ; H_a: XDouble Exp.

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
20	0.343	0.474	0.566	0.629	0.676
30	0.440	0.618	0.705	0.760	0.799
50	0.629	0.789	0.857	0.894	0.915

Table E.21 Power Study: Skewness Test – H_0: Weibull(1) ; H_a: XLogistic(0,1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
20	0.360	0.500	0.598	0.663	0.712
30	0.457	0.639	0.733	0.788	0.826
50	0.650	0.816	0.883	0.916	0.937

Table E.22 Power Study: Skewness Test – H_0: Weibull(1) ; H_a: XCauchy(0,1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
20	0.786	0.853	0.898	0.918	0.931
30	0.892	0.952	0.969	0.977	0.982
50	0.984	0.995	0.997	0.998	0.999

E.2.2 Kurtosis Test Results.

Table E.23 Power Study: Kurtosis Test – H_0: Weibull(1) ; H_a: Beta(2,2).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.004	0.023	0.053	0.088	0.125
15	0.012	0.081	0.180	0.277	0.363
25	0.061	0.284	0.507	0.656	0.759
50	0.599	0.930	0.983	0.994	0.998

Table E.24 Power Study: Kurtosis Test – H_0: Weibull(1) ; H_a: Beta(2,3).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.004	0.023	0.052	0.087	0.124
15	0.009	0.065	0.145	0.225	0.303
25	0.040	0.195	0.380	0.519	0.627
50	0.373	0.764	0.895	0.946	0.969

Table E.25 Power Study: Kurtosis Test – H_0: Weibull(1) ; H_a: Gamma(2,1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.006	0.031	0.066	0.106	0.147
15	0.006	0.038	0.084	0.134	0.184
25	0.010	0.053	0.116	0.175	0.231
50	0.024	0.094	0.165	0.227	0.283

Table E.26 Power Study: Kurtosis Test – H_0: Weibull(1) ; H_a: Normal(0,1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.003	0.018	0.043	0.071	0.103
15	0.003	0.027	0.067	0.114	0.165
25	0.009	0.064	0.155	0.250	0.339
50	0.073	0.321	0.519	0.645	0.732

Table E.27 Power Study: Kurtosis Test – H_0: Weibull(1) ; H_a: Unif(0,2).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
5	0.007	0.038	0.080	0.125	0.169
15	0.051	0.218	0.376	0.498	0.593
25	0.274	0.630	0.814	0.896	0.938
50	0.959	0.998	1.000	1.000	1.000

Table E.28 Power Study: Kurtosis Test – H_0: Weibull(1) ; H_a: Weibull(1).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
5	0.011	0.052	0.102	0.151	0.201
15	0.009	0.049	0.100	0.150	0.200
25	0.010	0.049	0.099	0.152	0.204
50	0.010	0.050	0.101	0.150	0.198

Table E.29 Power Study: Kurtosis Test – H_0: Weibull(1) ; H_a: Weibull(2,1,0).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
5	0.003	0.022	0.051	0.085	0.120
15	0.005	0.039	0.092	0.151	0.209
25	0.016	0.095	0.205	0.304	0.390
50	0.110	0.345	0.501	0.603	0.668

Table E.30 Power Study: Kurtosis Test – H_0: Weibull(1) ; H_a: Weibull(3.5,1,0).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
5	0.003	0.021	0.044	0.074	0.108
15	0.004	0.033	0.080	0.138	0.194
25	0.012	0.087	0.208	0.324	0.429
50	0.129	0.471	0.686	0.803	0.868

Table E.31 Power Study: Kurtosis Test – H_0: Weibull(1) ; H_a: $\chi^2(1)$.

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
20	0.027	0.088	0.149	0.203	0.256
30	0.026	0.088	0.149	0.207	0.263
50	0.028	0.094	0.166	0.227	0.283

Table E.32 Power Study: Kurtosis Test – H_0: Weibull(1) ; H_a: $\chi^2(4)$.

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
20	0.008	0.047	0.099	0.156	0.212
30	0.010	0.065	0.128	0.188	0.243
50	0.023	0.093	0.164	0.227	0.285

Table E.33 Power Study: Kurtosis Test – H_0: Weibull(1) ; H_a: Lognorm(0,1).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
20	0.067	0.148	0.213	0.271	0.326
30	0.085	0.179	0.257	0.321	0.378
50	0.117	0.236	0.330	0.399	0.454

Table E.34 Power Study: Kurtosis Test – H_0: Weibull(1) ; H_a: XDouble Exp.

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
20	0.329	0.441	0.513	0.571	0.626
30	0.403	0.546	0.641	0.699	0.742
50	0.555	0.722	0.802	0.846	0.876

Table E.35 Power Study: Kurtosis Test – H_0: Weibull(1) ; H_a: XLogistic(0,1).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
20	0.343	0.462	0.538	0.593	0.649
30	0.420	0.562	0.662	0.721	0.764
50	0.568	0.734	0.819	0.862	0.891

Table E.36 Power Study: Kurtosis Test – H_0: Weibull(1) ; H_a: XCauchy(0,1).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
20	0.776	0.830	0.860	0.886	0.913
30	0.867	0.923	0.952	0.963	0.970
50	0.968	0.988	0.994	0.995	0.997

E.3 H_0: Weibull($\beta = 1.5$)

E.3.1 Skewness Test Results.

Table E.37 Power Study: Skewness Test – H_0: Weibull(1.5) ; H_a: $\chi^2(4)$.

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
20	0.019	0.074	0.135	0.187	0.240
30	0.024	0.087	0.150	0.207	0.260
50	0.031	0.104	0.176	0.236	0.292

E.3.2 Kurtosis Test Results.

Table E.38 Power Study: Kurtosis Test – H_0: Weibull(1.5) ; H_a: $\chi^2(4)$.

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
20	0.019	0.076	0.132	0.187	0.241
30	0.021	0.078	0.140	0.196	0.248
50	0.028	0.090	0.157	0.218	0.271

E.4 H_0: Weibull($\beta = 3.5$)

E.4.1 Skewness Test Results.

Table E.39 Power Study: Skewness Test – H_0: Weibull(3.5) ; H_a: Beta(2,2).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.009	0.044	0.088	0.136	0.182
15	0.003	0.021	0.051	0.089	0.131
25	0.001	0.015	0.039	0.072	0.109
50	0.001	0.011	0.033	0.062	0.095

Table E.40 Power Study: Skewness Test – H_0: Weibull(3.5) ; H_a: Beta(2,3).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.010	0.049	0.098	0.145	0.193
15	0.007	0.043	0.094	0.148	0.201
25	0.009	0.053	0.111	0.173	0.231
50	0.013	0.086	0.171	0.248	0.320

Table E.41 Power Study: Skewness Test – H_0: Weibull(3.5) ; H_a: Gamma(2,1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.028	0.103	0.178	0.239	0.295
15	0.198	0.394	0.515	0.595	0.657
25	0.420	0.644	0.750	0.810	0.848
50	0.799	0.928	0.962	0.975	0.984

Table E.42 Power Study: Skewness Test – H_0: Weibull(3.5) ; H_a: Normal(0,1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.010	0.052	0.104	0.153	0.206
15	0.018	0.073	0.131	0.186	0.239
25	0.026	0.085	0.147	0.205	0.259
50	0.032	0.097	0.164	0.220	0.275

Table E.43 Power Study: Skewness Test – H_0: Weibull(3.5) ; H_a: Unif(0,2).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
5	0.012	0.054	0.101	0.145	0.189
15	0.002	0.018	0.045	0.080	0.118
25	0.001	0.013	0.036	0.068	0.102
50	0.001	0.009	0.029	0.056	0.089

Table E.44 Power Study: Skewness Test – H_0: Weibull(3.5) ; H_a: Weibull(1,1,0).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
5	0.061	0.166	0.252	0.318	0.373
15	0.363	0.601	0.714	0.781	0.827
25	0.677	0.857	0.917	0.946	0.961
50	0.967	0.995	0.998	0.999	1.000

Table E.45 Power Study: Skewness Test – H_0: Weibull(3.5) ; H_a: Weibull(2,1,0).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
5	0.014	0.063	0.119	0.171	0.222
15	0.042	0.138	0.222	0.293	0.358
25	0.091	0.231	0.336	0.419	0.484
50	0.220	0.454	0.581	0.660	0.719

Table E.46 Power Study: Skewness Test – H_0: Weibull(3.5) ; H_a: Weibull(3.5).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
5	0.010	0.050	0.102	0.152	0.202
15	0.010	0.050	0.099	0.151	0.203
25	0.010	0.052	0.102	0.153	0.202
50	0.010	0.048	0.098	0.148	0.200

Table E.47 Power Study: Skewness Test – H_0: Weibull(3.5) ; H_a: $\chi^2(1)$.

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
20	0.776	0.920	0.959	0.975	0.983
30	0.948	0.989	0.996	0.998	0.999
50	0.999	1.000	1.000	1.000	1.000

Table E.48 Power Study: Skewness Test – H_0: Weibull(3.5) ; H_a: $\chi^2(4)$.

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
20	0.309	0.528	0.645	0.715	0.765
30	0.509	0.726	0.825	0.872	0.901
50	0.800	0.931	0.963	0.976	0.983

Table E.49 Power Study: Skewness Test – H_0: Weibull(3.5) ; H_a: Lognorm(0,1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
20	0.763	0.896	0.940	0.959	0.971
30	0.932	0.979	0.991	0.995	0.997
50	0.997	1.000	1.000	1.000	1.000

Table E.50 Power Study: Skewness Test – H_0: Weibull(3.5) ; H_a: XDouble Exp.

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
20	0.926	0.968	0.981	0.986	0.989
30	0.989	0.996	0.998	0.999	0.999
50	1.000	1.000	1.000	1.000	1.000

Table E.51 Power Study: Skewness Test – H_0: Weibull(3.5) ; H_a: XLogistic(0,1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
20	0.958	0.988	0.995	0.997	0.998
30	0.997	0.999	1.000	1.000	1.000
50	1.000	1.000	1.000	1.000	1.000

Table E.52 Power Study: Skewness Test – H_0: Weibull(3.5) ; H_a: XCauchy(0,1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
20	0.986	0.994	0.997	0.998	0.998
30	0.999	1.000	1.000	1.000	1.000
50	1.000	1.000	1.000	1.000	1.000

E.4.2 Kurtosis Test Results.

Table E.53 Power Study: Kurtosis Test – H_0: Weibull(3.5) ; H_a: Beta(2,2).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.011	0.054	0.106	0.159	0.212
15	0.016	0.072	0.135	0.191	0.248
25	0.029	0.108	0.187	0.256	0.315
50	0.082	0.244	0.367	0.459	0.532

Table E.54 Power Study: Kurtosis Test – H_0: Weibull(3.5) ; H_a: Beta(2,3).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.011	0.056	0.110	0.166	0.216
15	0.015	0.067	0.125	0.178	0.230
25	0.020	0.079	0.140	0.200	0.256
50	0.035	0.125	0.207	0.279	0.340

Table E.55 Power Study: Kurtosis Test – H_0: Weibull(3.5) ; H_a: Gamma(2,1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.022	0.090	0.158	0.220	0.276
15	0.117	0.233	0.313	0.376	0.431
25	0.207	0.351	0.435	0.496	0.544
50	0.422	0.579	0.654	0.701	0.737

Table E.56 Power Study: Kurtosis Test – H_0: Weibull(3.5) ; H_a: Normal(0,1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.010	0.050	0.099	0.147	0.197
15	0.015	0.065	0.119	0.172	0.225
25	0.019	0.074	0.134	0.190	0.240
50	0.028	0.099	0.164	0.221	0.276

Table E.57 Power Study: Kurtosis Test – H_0: Weibull(3.5) ; H_a: Unif(0,2).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
5	0.019	0.078	0.145	0.206	0.262
15	0.064	0.189	0.291	0.371	0.436
25	0.170	0.378	0.505	0.593	0.654
50	0.545	0.792	0.878	0.918	0.942

Table E.58 Power Study: Kurtosis Test – H_0: Weibull(3.5) ; H_a: Weibull(1,1,0).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
5	0.045	0.143	0.222	0.289	0.349
15	0.203	0.351	0.438	0.498	0.548
25	0.355	0.522	0.605	0.657	0.697
50	0.661	0.789	0.840	0.870	0.890

Table E.59 Power Study: Kurtosis Test – H_0: Weibull(3.5) ; H_a: Weibull(2,1,0).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
5	0.013	0.061	0.116	0.171	0.221
15	0.030	0.096	0.162	0.219	0.274
25	0.046	0.129	0.198	0.260	0.314
50	0.086	0.186	0.265	0.326	0.382

Table E.60 Power Study: Kurtosis Test – H_0: Weibull(3.5) ; H_a: Weibull(3.5).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
5	0.010	0.051	0.099	0.151	0.201
15	0.010	0.052	0.102	0.149	0.202
25	0.009	0.051	0.100	0.151	0.200
50	0.010	0.049	0.098	0.148	0.200

Table E.61 Power Study: Kurtosis Test – H_0: Weibull(3.5) ; H_a: $\chi^2(1)$.

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
20	0.462	0.625	0.702	0.748	0.779
30	0.655	0.788	0.840	0.870	0.891
50	0.875	0.939	0.959	0.969	0.975

Table E.62 Power Study: Kurtosis Test – H_0: Weibull(3.5) ; H_a: $\chi^2(4)$.

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
20	0.163	0.294	0.380	0.441	0.493
30	0.251	0.398	0.483	0.540	0.587
50	0.424	0.574	0.647	0.695	0.732

Table E.63 Power Study: Kurtosis Test – H_0: Weibull(3.5) ; H_a: Lognorm(0,1).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
20	0.517	0.658	0.725	0.765	0.796
30	0.720	0.824	0.865	0.889	0.905
50	0.913	0.956	0.971	0.977	0.982

Table E.64 Power Study: Kurtosis Test – H_0: Weibull(3.5) ; H_a: XDouble Exp.

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
20	0.803	0.877	0.904	0.921	0.932
30	0.938	0.967	0.977	0.982	0.986
50	0.995	0.998	0.999	0.999	0.999

Table E.65 Power Study: Kurtosis Test – H_0: Weibull(3.5) ; H_a: XLogistic(0,1).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
20	0.829	0.902	0.928	0.942	0.952
30	0.953	0.978	0.985	0.989	0.991
50	0.997	0.999	0.999	1.000	1.000

Table E.66 Power Study: Kurtosis Test – H_0: Weibull(3.5) ; H_a: XCauchy(0,1).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
20	0.959	0.976	0.982	0.985	0.987
30	0.994	0.997	0.998	0.998	0.999
50	1.000	1.000	1.000	1.000	1.000

F.1 H_0: Weibull($\beta = 1$).

F.1.1 Lower Tail Skewness Test.

Table F.1 Power Study: Lower Tail Skewness Test – H_0: Weibull(1) ; H_a: Gamma(2,1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.018	0.083	0.152	0.217	0.280
15	0.040	0.138	0.233	0.314	0.382
25	0.061	0.180	0.284	0.368	0.441
50	0.099	0.251	0.368	0.455	0.526

Table F.2 Power Study: Lower Tail Skewness Test – H_0: Weibull(1) ; H_a: Normal(0,1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.061	0.222	0.337	0.435	0.518
15	0.471	0.708	0.812	0.866	0.900
25	0.753	0.898	0.943	0.964	0.976
50	0.973	0.993	0.997	0.999	0.999

Table F.3 Power Study: Lower Tail Skewness Test – H_0: Weibull(1) ; H_a: Weibull(3.5,1,0).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.058	0.216	0.332	0.430	0.514
15	0.464	0.715	0.827	0.883	0.915
25	0.774	0.920	0.960	0.976	0.985
50	0.983	0.998	0.999	1.000	1.000

Table F.4 Power Study: Lower Tail Skewness Test – H_0: Weibull(1) ; H_a: Weibull(1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.010	0.051	0.100	0.149	0.199
15	0.010	0.049	0.098	0.149	0.197
25	0.010	0.050	0.100	0.150	0.200
50	0.011	0.051	0.101	0.152	0.201

F.1.2 Upper Tail Skewness Test.

Table F.5 Power Study: Upper Tail Skewness Test – H_0: Weibull(1) ; H_a: Lognorm(0,1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
20	0.095	0.209	0.309	0.386	0.450
30	0.124	0.271	0.381	0.464	0.535
50	0.178	0.365	0.486	0.571	0.638

Table F.6 Power Study: Upper Tail Skewness Test – H_0: Weibull(1) ; H_a: XLogistic(0,1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
20	0.411	0.599	0.710	0.775	0.821
30	0.527	0.736	0.826	0.876	0.908
50	0.717	0.883	0.935	0.958	0.972

Table F.7 Power Study: Upper Tail Skewness Test – H_0: Weibull(1) ; H_a: Weibull(1).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.010	0.049	0.098	0.149	0.201
15	0.010	0.051	0.102	0.151	0.203
25	0.011	0.051	0.100	0.150	0.201
50	0.010	0.052	0.106	0.155	0.203

F.1.3 Lower Tail Kurtosis Test.

Table F.8 Power Study: Lower Tail Kurtosis Test – H_0: Weibull(1) ; H_a: Normal(0,1).

Sample	Significance Level (α)				
Size	0.01	0.05	0.10	0.15	0.20
5	0.005	0.035	0.078	0.129	0.183
15	0.008	0.067	0.164	0.271	0.385
25	0.020	0.155	0.338	0.495	0.620
50	0.142	0.519	0.732	0.842	0.902

Table F.9 Power Study: Lower Tail Kurtosis Test – H_0: Weibull(1) ; H_a: Unif(0,2).

Sample	Significance Level (α)				
Size	0.01	0.05	0.10	0.15	0.20
5	0.013	0.069	0.144	0.217	0.286
15	0.098	0.377	0.593	0.734	0.828
25	0.407	0.817	0.941	0.976	0.990
50	0.986	1.000	1.000	1.000	1.000

Table F.10 Power Study: Lower Tail Kurtosis Test – H_0: Weibull(1) ; H_a: Weibull(3.5,1,0).

Sample	Significance Level (α)				
Size	0.01	0.05	0.10	0.15	0.20
5	0.005	0.038	0.084	0.137	0.195
15	0.010	0.079	0.195	0.314	0.437
25	0.028	0.211	0.435	0.598	0.722
50	0.237	0.682	0.869	0.939	0.970

Table F.11 Power Study: Lower Tail Kurtosis Test – H_0: Weibull(1) ; H_a: Weibull(1).

Sample	Significance Level (α)				
Size	0.01	0.05	0.10	0.15	0.20
5	0.011	0.049	0.098	0.149	0.199
15	0.009	0.052	0.100	0.147	0.197
25	0.010	0.049	0.100	0.149	0.199
50	0.011	0.050	0.101	0.150	0.200

F.1.4 Upper Tail Kurtosis Test.

Table F.12 Power Study: Upper Tail Kurtosis Test – H_0: Weibull(1) ; H_a: Lognormal(0,1).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
20	0.090	0.194	0.286	0.360	0.425
30	0.114	0.245	0.349	0.429	0.498
50	0.156	0.324	0.439	0.524	0.593

Table F.13 Power Study: Upper Tail Kurtosis Test – H_0: Weibull(1) ; H_a: XLogistic(0,1).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
20	0.389	0.534	0.642	0.718	0.768
30	0.474	0.669	0.766	0.824	0.864
50	0.639	0.819	0.889	0.924	0.947

Table F.14 Power Study: Upper Tail Kurtosis Test – H_0: Weibull(1) ; H_a: Weibull(1).

Sample Size	Significance Level (α) 0.01	0.05	0.10	0.15	0.20
5	0.009	0.049	0.101	0.153	0.204
15	0.010	0.050	0.100	0.151	0.203
25	0.011	0.052	0.104	0.151	0.202
50	0.010	0.051	0.101	0.150	0.199

F.2 H_0: Weibull($\beta = 3.5$).

F.2.1 Upper Tail Skewness Test.

Table F.15 Power Study: Upper Tail Skewness Test – H_0: Weibull(3.5) ; H_a: Beta(2,3).

Sample	Significance Level (α)				
Size	0.01	0.05	0.10	0.15	0.20
5	0.013	0.063	0.126	0.187	0.245
15	0.015	0.087	0.178	0.261	0.339
25	0.020	0.112	0.223	0.322	0.409
50	0.030	0.172	0.320	0.438	0.540

Table F.16 Power Study: Upper Tail Skewness Test – H_0: Weibull(3.5) ; H_a: Weibull(1,1,0).

Sample	Significance Level (α)				
Size	0.01	0.05	0.10	0.15	0.20
5	0.091	0.246	0.360	0.447	0.522
15	0.461	0.716	0.828	0.882	0.918
25	0.757	0.917	0.962	0.979	0.987
50	0.981	0.998	0.999	1.000	1.000

Table F.17 Power Study: Upper Tail Skewness Test – H_0: Weibull(3.5) ; H_a: Weibull(2,1,0).

Sample	Significance Level (α)				
Size	0.01	0.05	0.10	0.15	0.20
5	0.018	0.088	0.163	0.231	0.296
15	0.072	0.216	0.338	0.435	0.512
25	0.139	0.339	0.483	0.581	0.658
50	0.310	0.583	0.723	0.800	0.850

F.2.2 Lower Tail Kurtosis Test.

Table F.18 Power Study: Lower Tail Kurtosis Test – H_0: Weibull(3.5) ; H_a: Beta(2,2).

Sample	Significance Level (α)				
Size	0.01	0.05	0.10	0.15	0.20
5	0.013	0.062	0.120	0.178	0.236
15	0.030	0.122	0.216	0.298	0.372
25	0.050	0.182	0.308	0.405	0.488
50	0.133	0.368	0.536	0.644	0.722

Table F.19 Power Study: Lower Tail Kurtosis Test – H_0: Weibull(3.5) ; H_a: Unif(0,2).

Sample	Significance Level (α)				
Size	0.01	0.05	0.10	0.15	0.20
5	0.022	0.093	0.167	0.228	0.290
15	0.105	0.285	0.420	0.517	0.595
25	0.245	0.516	0.663	0.752	0.813
50	0.658	0.877	0.940	0.966	0.979

F.2.3 Upper Tail Kurtosis Test.

Table F.20 Power Study: Upper Tail Kurtosis Test – H_0: Weibull(3.5) ; H_a: Weibull(1,1,0).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.056	0.158	0.236	0.292	0.335
15	0.248	0.400	0.489	0.547	0.593
25	0.418	0.590	0.671	0.720	0.758
50	0.715	0.837	0.884	0.909	0.927

Table F.21 Power Study: Upper Tail Kurtosis Test – H_0: Weibull(3.5) ; H_a: Weibull(2,1,0).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.013	0.058	0.113	0.167	0.216
15	0.039	0.107	0.168	0.219	0.266
25	0.058	0.144	0.210	0.262	0.308
50	0.104	0.214	0.290	0.349	0.399

Table F.22 Power Study: Upper Tail Kurtosis Test – H_0: Weibull(3.5) ; H_a: Weibull(3.5).

Sample Size	Significance Level (α)				
	0.01	0.05	0.10	0.15	0.20
5	0.009	0.049	0.100	0.152	0.204
15	0.011	0.050	0.103	0.154	0.203
25	0.009	0.051	0.100	0.149	0.200
50	0.010	0.051	0.103	0.153	0.203

Appendix G. Power of One-Sided Variants of the Sequential Test and Individual Tests

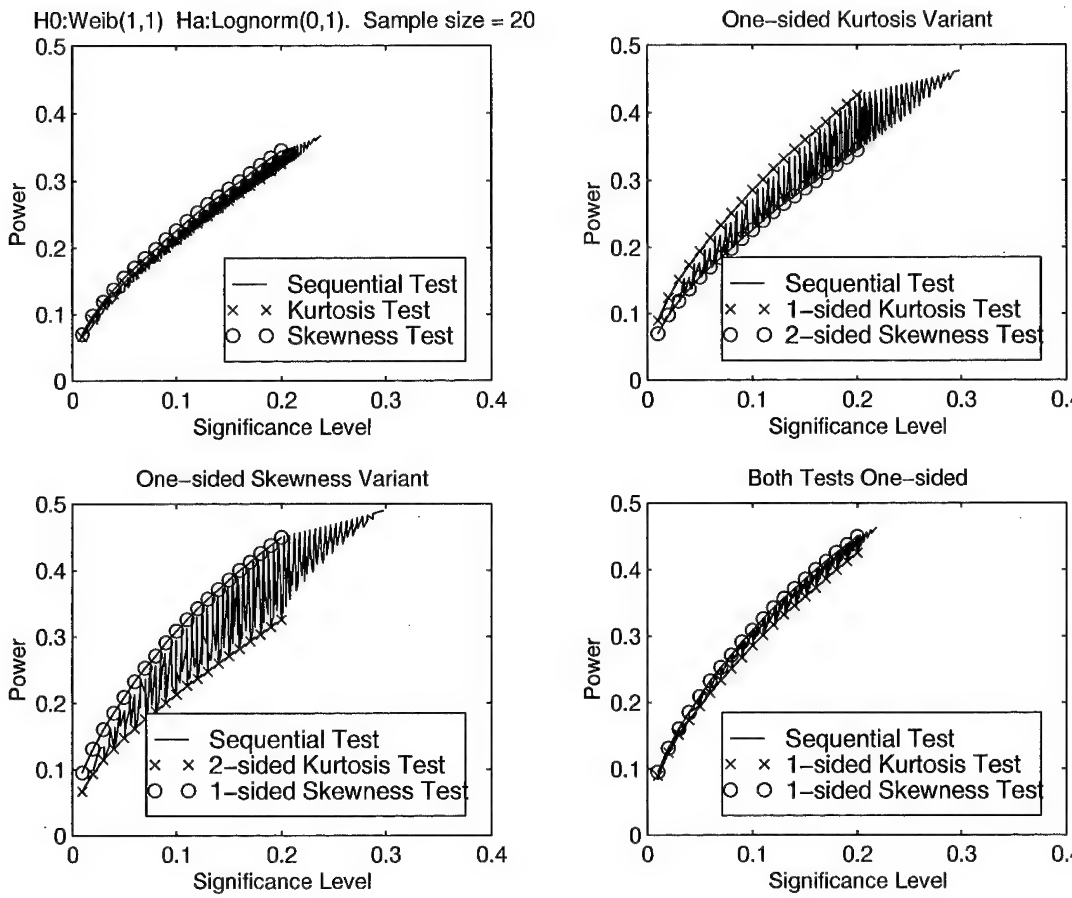


Figure G.1 Power of One-Sided Variants: H_0: Weibull($\beta = 1$); H_a: Lognormal(0,1), $n = 20$.

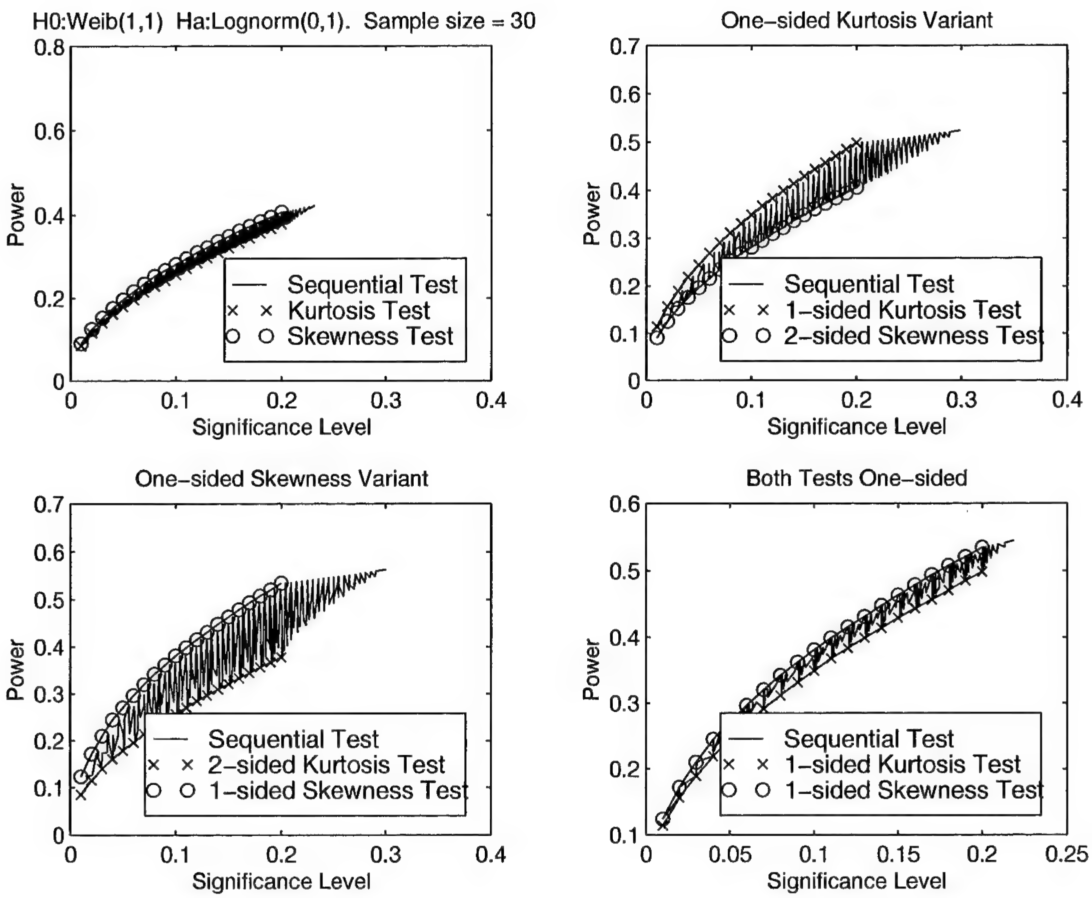


Figure G.2 Power of One-Sided Variants: H_0: Weibull($\beta = 1$); H_a: Lognormal(0,1), $n = 30$.

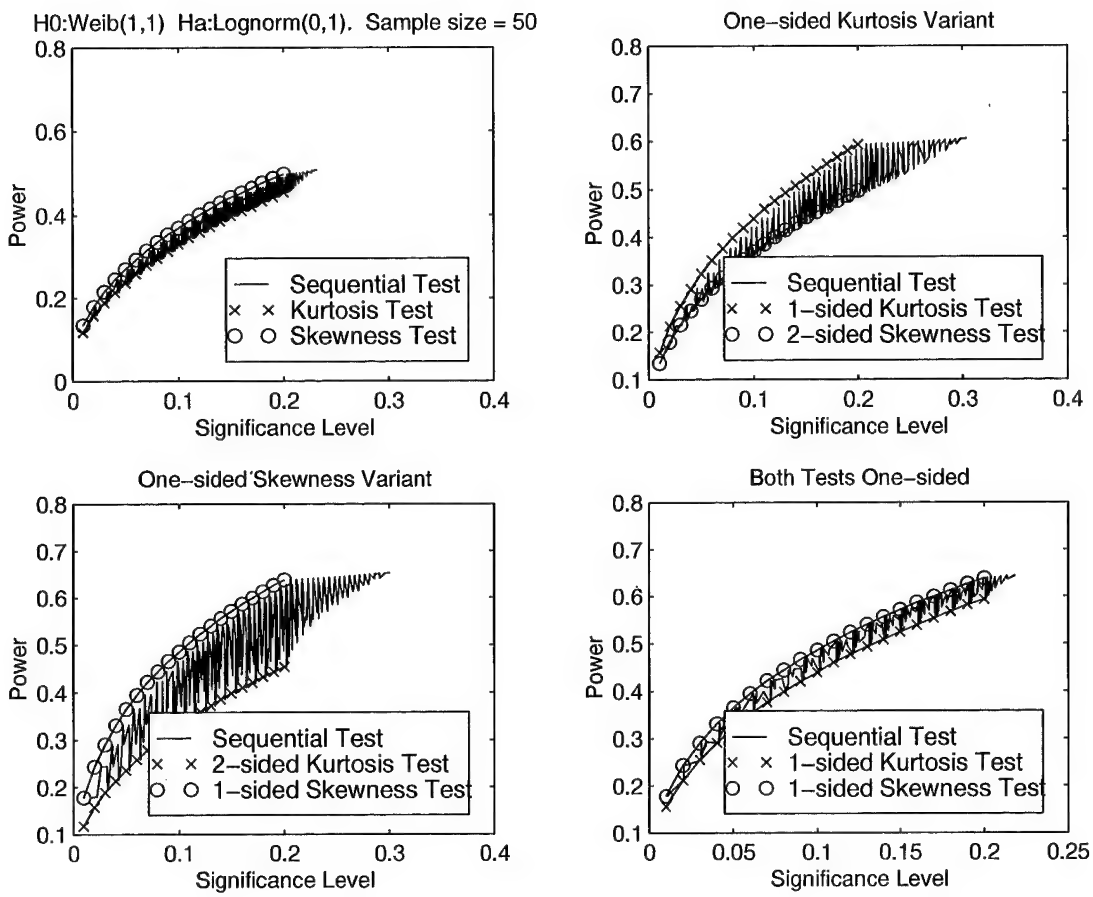


Figure G.3 Power of One-Sided Variants: H_0: Weibull($\beta = 1$); H_a: Lognormal(0,1), $n = 50$.

Appendix H. MATLAB Code

H.1 Critical Values Generator

```
%========================================================================
% Critical Values Generator
% Jonathan Clough    22 Oct 97
% Generates critical values for sample skewness and sample kurtosis
% of Weibull samples of sizes n=5(5)50 with known shape via Monte Carlo
% Simulation with 100,000 trials
%========================================================================

% Initialize values for 3-parameter Weibull
theta = 1; % (scale)
beta  = 1; % (shape)                 % substitute desired shape here
delta = 0; % (location)

% Convert to a and b params used by weibrnd function
a = 1/(theta^beta);
b = beta;

numsamples = 100000;                     % Number of trials of size n to generate
                                         % Make an array of indicies (icount)
icount = [1:numsamples];                 % to calculate median rank
mrank = (icount-0.3)./(numsamples+0.4);  % Calculate the median ranks array

% Begin main loop over sample sizes
% Generate numsamples of size n and find sample skewness and kurtosis
% for each one; Save values in the arrays skew() and kurt().

for n = 5:5:50
   sprintf('Starting sample size n = %d\n',n)
   skew = zeros(1,numsamples);           % Preallocate the vectors to expedite
   kurt=zeros(1,numsamples);

   for i=1:numsamples
     x = weibrnd(a,b,1,n)+delta;
     sk = skewness(x);                   % Need to pull scalar values
     kt = kurtosis(x);                   % out or it gets confused
     skew(i) = sk;
     kurt(i) = kt;
     if rem(i,1000) == 0 disp(i); end    % Helps me see progress
   end % for

   disp('finished generating')

% Sort the arrays in ascending order to facilitate interpolating critvals

   skew = sort(skew);
   kurt = sort(kurt);
```

```
   disp('sorted')

% For the given sample size n, find upper and lower tail critical values
% by iterpolation.  Step through alpha=0.005(0.005)0.10 and 0.10(0.10)0.20.

   alpha = 0.005;                           % Intialize
   step  = 0.005;

   while alpha < 0.21                       % Matlab wouldn't let me use <=0.20

      if alpha <= 0.10                      % Find column index for critvals arrays
         j = round(alpha*200);              % maps [0.005,0.10] to [1,20]
      else
         j = round((alpha+0.10)*100);       % maps [0.11,0.20] to [21,30]
      end % if

      i = round(n/5);                       % Find row index for critvals arrays

      skewuptail(i,j)=interp1(mrank,skew,1-alpha);   % Interpolate to find
      skewlotail(i,j)=interp1(mrank,skew,alpha);     % the critical values
      kurtuptail(i,j)=interp1(mrank,kurt,1-alpha);   % and save them in arrays
      kurtlotail(i,j)=interp1(mrank,kurt,alpha);     % indexed by sample size
     % and alpha

      if alpha >= 0.10
         step = 0.01;                       % Change the step after 0.10
      end %if

      alpha = alpha + step;                 % Increment alpha for the next level

   end % while

   sprintf('Done with sample size %d\n',n)

end % for


% End of loops that calculate the arrays of critical values

disp('Saving and writing to files')
save skewcrit skewuptail skewlotail;  % Save the crit vals to binary files for
save kurtcrit kurtuptail kurtlotail;  % use in the subsequent studies

format = '%2d    %4.3f    %4.3f    %4.3f    %4.3f    %4.3f    %4.3f    %4.3f
       %4.3f    %4.3f    %4.3f    %4.3f    %4.3f    %4.3f    %4.3f    %4.3f
       %4.3f    %4.3f    %4.3f    %4.3f    %4.3f    %4.3f    %4.3f    %4.3f
       %4.3f    %4.3f    %4.3f    %4.3f    %4.3f    %4.3f    %4.3f\n';

col = (5:5:50); % vector to keep track of sample sizes in output

% Send tables to 4 files  (upper/lower skewness/kurtosis)
```

```
table = [col',skewuptail];
file1 = fopen('skewup.txt','w');
fprintf(file1,'Critical Values for Skewness -- Upper Tail\n');
fprintf(file1,'Shape = %d\n',beta);
fprintf(file1,'\n');
fprintf(file1,format,table');
fclose(file1);

table=[col',skewlotail];
file2 = fopen('skewlo.txt','w');
fprintf(file2,'Critical Values for Skewness -- Lower Tail\n');
fprintf(file2,'Shape = %d\n',beta);
fprintf(file2,'\n');
fprintf(file2,format,table');
fclose(file2);

table=[col',kurtuptail];
file3 = fopen('kurtup.txt','w');
fprintf(file3,'Critical Values for Kurtosis -- Upper Tail\n');
fprintf(file3,'Shape = %d\n',beta);
fprintf(file3,'\n');
fprintf(file3,format,table');
fclose(file3);

table=[col',kurtlotail];
file4 = fopen('kurtlo.txt','w');
fprintf(file4,'Critical Values for Kurtosis -- Lower Tail\n');
fprintf(file4,'Shape = %d\n',beta);
fprintf(file4,'\n');
fprintf(file4,format,table');
fclose(file4);
```

H.2 Attained Significance Levels Generator

```
%=======================================================================
% Attained Significance Levels Generator
% Jonathan Clough  4 Nov 97   Revised 6 Nov 97
% Computes the attained significance levels for a sequential
% goodness-of-fit test for the Weibull distribution using two-sided
% tests for skewness and kurtosis.
%=======================================================================


% Initialize values for a 3-parameter Weibull
theta = 1;  % scale
beta  = 1;  % shape      (will be varied)
delta = 0;  % location

% Convert parameters to form used by MATLABs weibrnd function
a = 1/(theta^beta);
b = beta;

numsamples = 100000;  % number of samples of size n to generate

load kurtcrits1;      % Load the critical values arrays generated
load skewcrits1;      % from critvals. (Need to change names for diff shapes)
                      % This will load the following arrays:
                      % kurtlotail   kurtuptail
                      % skewlotail   skewuptail

A = zeros(20,20,10);   % Initialize the counter array to all 0s

for n = 5:5:50
   sprintf('Starting sample size %d',n)

  for i = 1:numsamples
     x = weibrnd(a,b,1,n)+delta;   % Generates a 1xn vector of Weibull deviates
     sk = skewness(x);
     kt = kurtosis(x);
     if rem(i,1000) == 0 disp(i); end        % Helps me see progress

     % Initialize the placeholders for this particular sample
     icurr = 1;  jcurr = 1;
     istop = 21; jstop = 21;

     % Conduct the Skewness Test (Test #1) at all alpha levels until
     % you find a failure, then save that point in istop.
     % We'll use alpha levels from 0.01 to 0.20.  The corresponding
     % crit values for the two sided test are alpha/2  and  1-(alpha/2)
     % and correspond to columns 1 to 20 respectively in the upper and
     % lower tail arrays loaded earlier.  Columns 21-30 in these arrays
     % are not needed at this time.
```

```
    while icurr < istop
      if sk < skewlotail(n/5,icurr)          % fail lower tail
         istop = icurr;
      elseif sk > skewuptail(n/5,icurr)      % fail upper tail
         istop = icurr;
      end % if

      icurr = icurr + 1;               % Increment icurr if passed or failed.
                                       % If it failed, the resetting of istop
    end % while                        % will force loop termination.
                        % When the while loop ends, istop will
                                       % equal the failure point (1-20) or
                                       % equal 21 if it passed all levels.

    % Now conduct the Kurtosis Test (Test #2) similarly

    while jcurr < jstop
      if kt < kurtlotail(n/5,jcurr)      % fail lower tail
        jstop = jcurr;
      elseif kt > kurtuptail(n/5,jcurr)  % fail upper tail
        jstop = jcurr;
      end % if

      jcurr = jcurr + 1;
                                        % At this point jstop will hold the
    end % while                         % fail point (1-20) if it failed;
                                        % If it passed all, then jstop = 21.

    % Now figure out which cells to increment in the counter array.
    % Fail1 is an array of 0s and 1s indicating what levels the
    % sample failed Test#1; Fail2 is the similar array for Test #2.
    % The Inc array is the union of the two and will be used to increment
    % the main count array A.

    Fail1 = zeros(20,20);               % Initialize them to all 0s
    Fail2 = zeros(20,20);
    Inc   = zeros(20,20);

    if istop < 21  Fail1(istop:20,:) = 1; end   % Fill in 1s where failed
    if jstop < 21  Fail2(:,jstop:20) = 1; end   % unless failed none

    Inc = Fail1 | Fail2;
    A(:,:,n/5) = A(:,:,n/5) + Inc;

  end % for loop (i) -- go back for next sample of size n

  sprintf('Finished sample size %d -- going to next.\n',n);

end % for loop (n) -- now change sample sizes

                          % Now A has counts for all sample sizes (5:5:50)
```

```
A = A./numsamples;      % Divide by numsamples to get alpha levels

% Now output to a file

save sigtable1 A;       % Save A to a binary file for future use
file1 = fopen('sigtabl1.txt','w');  % And save to a text file

format = '%4.3f & %4.3f & %4.3f & %4.3f & %4.3f & %4.3f & %4.3f & %4.3f &
  %4.3f & %4.3f & %4.3f & %4.3f & %4.3f & %4.3f & %4.3f & %4.3f & %4.3f &
  %4.3f & %4.3f & %4.3f\\\\ \\hline \n';

fprintf(file1,'Table for Shape = %d \n',beta);
for n=5:5:50
  fprintf(file1,'Significance levels for sample size %2d \n',n);
  fprintf(file1,format,A(:,:,n/5)');
  fprintf(file1,'\n \n');
end % for

fclose(file1);
disp('Done: File saved.')
```

```
%===========================================================================
% Power Study  -- H0: Weib(1,1)   H1: Beta(2,2)
% ------------------------------------------
% Jonathan Clough  14 Nov 97
% Computes the power of the sequential goodness-of-fit test for the
% Weibull distribution based on two-sided tests for skewness and
% kurtosis at each combination of significance levels between 0.01(0.01)0.20
% and sample sizes 5,15,25, 50 against the H1 noted above. Used to compare
% against Bush's results.
%===========================================================================

%  Parameters for the alternate distribution [ Beta(2,2) ]
a = 2;
b = 2;

numsamples = 40000;       % number of samples of size n to generate

load kurtcrits1;          % Load the critical values arrays generated for
load skewcrits1;          % H0:Weib(1,1) from the crit vals simulation.
                          % (Need to change names for diff shapes)
                          % This will load the following arrays:
                          % kurtlotail   kurtuptail
                          % skewlotail   skewuptail

A = zeros(20,20,10);      % Initialize the failure counter array to all 0s

n = 5;                    % 1st Sample size
step = 10;                % Start with a step of 10 to do 5,15, and 25

while n < 51              % Cycle until you step past sample size 50

   sprintf('Starting sample size %d',n)

  for i = 1:numsamples
     x  = betarnd(a,b,1,n);    % Generates a 1xn vector of Beta(2,2) deviates
     sk = skewness(x);
     kt = kurtosis(x);
     if rem(i,1000) == 0 disp(i); end        % Helps me see progress

     % Initialize the placeholders for this particular sample
     icurr = 1;  jcurr = 1;
     istop = 21; jstop = 21;

     % Conduct the Skewness Test (Test #1) at all alpha levels until
     % you find a failure, then save that point in istop.
     % We'll use alpha levels from 0.01 to 0.20.  The corresponding
     % crit values for the two sided test are alpha/2  and  1-(alpha/2)
     % and correspond to columns 1 to 20 respectively in the upper and
     % lower tail arrays loaded earlier.  Columns 21-30 in these arrays
```

```
  % are not needed at this time.

  while icurr < istop
    if sk < skewlotail(n/5,icurr)         % fail lower tail
       istop = icurr;
    elseif sk > skewuptail(n/5,icurr)     % fail upper tail
       istop = icurr;
    end % if

    icurr = icurr + 1;               % Increment icurr if passed or failed.
                                     % If it failed, the resetting of istop
  end % while                        % will force loop termination.
                                     % When the while loop ends, istop will
                                     % equal the failure point (1-20) or
                                     % equal 21 if it passed all levels.

  % Now conduct the Kurtosis Test (Test #2) similarly

  while jcurr < jstop
    if kt < kurtlotail(n/5,jcurr)         % fail lower tail
      jstop = jcurr;
    elseif kt > kurtuptail(n/5,jcurr)     % fail upper tail
      jstop = jcurr;
    end % if

    jcurr = jcurr + 1;

  end % while                        % At this point jstop will hold the fail
                                     % point (1-20) if it failed; If it passed
                                     % all the jstop=21

  % Now figure out which cells to increment in the counter array.
  % Fail1 is an array of 0s and 1s indicating what levels the
  % sample failed Test#1; Fail2 is the similar array for Test #2.
  % The Inc array is the union of the two and will be used to increment
  % the main count array A.

  Fail1 = zeros(20,20);              % Initialize them to all 0s
  Fail2 = zeros(20,20);
  Inc   = zeros(20,20);

  if istop < 21  Fail1(istop:20,:) = 1; end   % Fill in 1s where failed
  if jstop < 21  Fail2(:,jstop:20) = 1; end   % unless failed none

  Inc = Fail1 | Fail2;
  A(:,:,n/5) = A(:,:,n/5) + Inc;

end % for loop (i) -- go back for next sample of size n

sprintf('Finished sample size %d -- going to next.\n',n);
```

```
  if n == 25
     step = 25;         % Once you hit size 25, jump to 50 on the next iteration
  end %

  n = n + step;         % increment n for the next sample size

end % while loop (n) -- now change sample sizes

                        % Now A has counts for all sample sizes (5:5:50)
A = A./numsamples;      % Divide by numsamples to get alpha levels

% Now output to a file

save pwrbeta22 A;       % Save A to a binary file for future use (eg. plots)

file1 = fopen('pwrbeta22.txt','w');  % And save to a text file

cols=[0.01:0.01:0.2];               % labels for leftmost column

format = '& %3.2f & %4.3f & %4.3f & %4.3f & %4.3f & %4.3f & %4.3f & %4.3f
  & %4.3f & %4.3f & %4.3f & %4.3f & %4.3f & %4.3f & %4.3f & %4.3f & %4.3f
  & %4.3f & %4.3f & %4.3f & %4.3f\\\\ \\cline{2-22} \n';

heading = '  & 0.01 & 0.02 & 0.03 & 0.04 & 0.05 & 0.06 & 0.07 & 0.08 & 0.09
      & 0.10 & 0.11 & 0.12 & 0.13 & 0.14 & 0.15 & 0.16 & 0.17 & 0.18 & 0.19
      & 0.20 \\\\ \\hline \\hline \n';

fprintf(file1,'Power Study: H0: Weib(1,1)  H1: Beta(2,2) \n');
for n=5:5:50
  fprintf(file1,'Powers for sample size %2d \n',n);
  fprintf(file1,heading);
  table=[cols',A(:,:,n/5)];
  fprintf(file1,format,table');
  fprintf(file1,'\n \n');
end % for

fclose(file1);
disp('Done: File saved.')
```

H.4 Individual One-Tailed Test Power Study – Kurtosis Example

Because the code for the individual two-tailed tests is nearly identical to that in the previous section (except for the omission one of the tests), it is not reproduced here. The one-tailed versions of the tests, however, required some code modifications. An example for the lower-tail kurtosis test is given below.

```
%========================================================================
% Power Study: Kurtosis Test Only -- H0: Weib(1,1)    H1: Norm(0,1)
% One Tailed Version -- Lower Tail
% --------------------------------------------
% Jonathan Clough  11 Dec 97
% Computes the power of the goodness-of-fit test for the
% Weibull distribution based on two-sided tests for kurtosis
% at significance levels between 0.01(0.01)0.20 and sample sizes
% 5,15,25, 50 against the H1 noted above. Used to compare
% against Bush's results.
%========================================================================

% Parameters for Alternate distribution: Norm(0,1)
mu    = 0;
sigma = 1;

numsamples = 40000;  % number of samples of size n to generate

rand('seed',32539);  % Change this for each run

load kurtcrits1;     % Load the critical values arrays generated for H0:Weib(1,1)
                     % This will load the following arrays:
                     % kurtlotail   kurtuptail

A = zeros(10,20);    % Initialize the failure counter array to all 0s
                     % 1st index (rows) tracks sample sizes (5:5:50)
                     % 2nd index (cols) tracks alpha levels (.01:.01:.20)

n = 5;               % 1st Sample size
step = 10;           % Start with a step of 10 to do 5,15, and 25

while n < 51         % Cycle until you step past sample size 50

   sprintf('Starting sample size %d',n)

  for i = 1:numsamples
     x = normrnd(mu,sigma,1,n);   % Generates a 1xn vector of Normal deviates
     kt = kurtosis(x);
     if rem(i,1000) == 0 disp(i); end        % Helps me see progress
```

```
  % Initialize the placeholders for this particular sample
  jcurr = 1;
  jstop = 21;
  jindex = 0;     % Added this for the one-tailed test to help ref the
                  % proper critical values -- Now, jindex = jcurr*2 if
                  % jcurr<= 10 and jindex = jcurr+10 if jcurr > 10.
                  % This is due to the fact that the critvals arrays
                  % columns are indexed by 1-30 where the ith column
                  % is for the .01*i alpha level of the two tailed test
                  % meaning the actual value is the .01*i/2 crit value.
                  % For the one tailed test we want the actual .01*i
                  % critical value, so we must change the reference.

  % Now conduct the Kurtosis Test (1-tailed)

  while jcurr < jstop

    if jcurr <= 10           % This new block assigns jindex correctly
       jindex = jcurr*2;
    else
       jindex = jcurr +10;
    end;

    if kt < kurtlotail(n/5,jindex)      % fail lower tail
       jstop = jcurr;
    end % if

    jcurr = jcurr + 1;

  end % while

  % At this point jstop will hold the fail point (1-20) if it failed;
  % If it passed all then jstop = 21.

  % Now figure out which cells to increment in the counter array.
  % Fail is an array of 0s and 1s indicating what levels the
  % sample failed the Kurtosis test.  It is used to increment the
  % counter array A.

  Fail = zeros(1,20);                         % Initialize it to all 0s

  if jstop < 21  Fail(1,jstop:20) = 1; end   % Switch to 1s for all failures

  A(n/5,:) = A(n/5,:) + Fail;

end % for loop (i) -- go back for next sample of size n

sprintf('Finished sample size %d -- going to next.\n',n);

if n == 25
  step = 25;       % Once you hit size 25, jump to 50 on the next iteration
```

```
  end %

  n = n + step;       % increment n for the next sample size

end % while loop (n) -- now change sample sizes

                        % Now A has counts for all sample sizes (5:5:50)
A = A./numsamples;      % Divide by numsamples to get alpha levels

% Now output to a file

save pwr1norm A;       % Save A to a binary file for future ref

file1 = fopen('pwr1norm.txt','w');  % And save to a text file


format = ' %2d & %4.3f & %4.3f & %4.3f & %4.3f & %4.3f \\\\  \n';

heading = 'Size & 0.01 & 0.05 & 0.10 & 0.15 & 0.20 \\\\ \\hline \\hline \n';

fprintf(file1,'Power Study: Lower Tail Kurtosis Test -- H0: Weib(1,1)
        H1: Normal(0,1) \n');
fprintf(file1,heading);

for n=5:5:50
  row=[n,A(n/5,[1 5 10 15 20])];
  fprintf(file1,format,row);
  fprintf(file1,'\n \n');
end % for

fclose(file1);
disp('Done: File saved.')
```

H.5 Miscellaneous Code

H.5.1 Contour Plots for Attained Significance Levels.

```
%=================================================================
% Generate contour charts of Attained Sig Levels for Sequential test
% Jonathan Clough   10 Dec 97
%=================================================================

load sigtable1;

x = 0.01:.01:.20;                    % vectors for labeling the axes
y = x;
v = .01:.01:.30;                     % vector for desired contour levels
S10=A(:,:,2);                        % Pick off the sample size 10 layer
[s10label h]=contour(x,y,S10,v);     % Get the labeling structure
clabel(s10label,h,'manual');
labels = ['0.01';
          '    ';
          '    ';
          '    ';
          '0.05';
          '    ';
          '    ';
          '    ';
          '    ';
          '0.10';
          '    ';
          '    ';
          '    ';
          '    ';
          '0.15';
          '    ';
          '    ';
          '    ';
          '    ';
          '0.20';];
set(gca,'XTick',x)                   % Adjust the x-axis tick marks
set(gca,'YTick',y)                   %            y-
set(gca,'XTickLabel',labels)         % And relabel them both
set(gca,'YTickLabel',labels)
title('Attained Significance Levels: \beta = 1, Sample size = 10');
grid;
xlabel('Kurtosis Test  \alpha-level');
ylabel('Skewness Test  \alpha-level');
```

H.5.2 Random Variate Generators.

```
function y = logisticrnd(a,b,n);
% LOGISTICRND returns a 1xn vector of Logistic(a,b) random variates, where
% a is the location paramter and b is the scale parameter.  The function
% employs the inverse transform method and thus is exact.

u = unifrnd(0,1,1,n);

y = -b.*log(1./u -1) + a;


function y = doublexprnd(n);
% DOUBLEXPRND returns a 1xn vector of Double Exponential(0,1) random variates
% This function draws a 1xn vector of Uniform(0,1), splits it into two subvectors
% in order to execute the piecewise inverse transform, then merges it back

u = unifrnd(0,1,1,n);
low  = (u<=0.5);           % 1xn vectors of 0 and 1 according to the condition
high = (u>0.5);

low  = low.*u;             % Now they are 1xn vectors of 0 and elements of u
high = high.*u;

u1 = low(find(low));       % Assigns only the nonzero elements to u1 or u2
u2 = high(find(high));

y1 = log(2.*u1);           % Make the appropriate inverse transform
y2 = -log(2-2.*u2);

y = [y1 y2];               % Merge the two parts back together


function y = cauchyrnd(a,b,n);
% CAUCHYRND returns a 1xn vector of Cauchy(a,b) random variates, where a is
% the location parameter and b is the scale parameter.  The function employs the
%i nverse transform method and thus is exact.

u = unifrnd(0,1,1,n);

y = b.*tan(pi.*(u-0.5))+a;
```

```
%===========================================================================
% Power Plot Generator
% Jonathan Clough           16 Dec 97
% This file creates plots comparing the power of the sequential test
% with the individual two-sided tests against a specific alternative
% for a given sample size
%===========================================================================

% ----------------- Sample Size 5 ---------------------------------------
   % Get Data for Power of the Sequential Test

subplot(2,2,1)
load ../../../shape1/sigtable1;
Level=A(:,:,1);                % Load the sig levels of the sample size
clear A;
load ../../../powerstudy/weib1/unif/pwrunif;
Power = A(:,:,1);              % Load the power for the same sample size
clear A;
Power=Power(:);                % String out the arrays into vectors
Level=Level(:);
[sLevel i] = sort(Level);      % Sort the levels and save the indicies to
sPower = Power(i);             % sort the corresponding powers pairwise

   % Data for the Power for the 2-sided Kurtosis Test

load ../../../powerstudy2/weib1/kurt/unif/pwr2unif;
Kpower = A(1,:);               % Adjust 1st index acc to sample size
clear A;
Klevel = .01:.01:.20;          % Vector of corresonding sig levels

   % Data for the Power for the 2-sided Skewness Test

load ../../../powerstudy2/weib1/skew/unif/pwr2unif;
SKpower = A(1,:);               % Adjust 1st index acc to sample size
clear A;

plot(sLevel, sPower,Klevel, Kpower,'gx',Klevel, SKpower,'ro',Klevel,
     Kpower,'g',Klevel, SKpower,'r')
title('H0:Weib(1,1)  Ha:Unif(0,2).  Sample size = 5')
xlabel('Significance Level')
ylabel('Power')
%legend('Sequential Test','Kurtosis Test','Skewness Test',4)

% ------------------------ Sample Size 15 -------------------------------

   % Get Data for Power of the Sequential Test

subplot(2,2,2)
load ../../../shape1/sigtable1;
```

```
Level=A(:,:,3);                 % Load the sig levels of the sample size
clear A;
load ../../../powerstudy/weib1/unif/pwrunif;
Power = A(:,:,3);               % Load the power for the same sample size
clear A;
Power=Power(:);                 % String out the arrays into vectors
Level=Level(:);
[sLevel i] = sort(Level);       % Sort the levels and save the indicies to
sPower = Power(i);              % sort the corresponding powers pairwise

   % Data for the Power for the 2-sided Kurtosis Test

load ../../../powerstudy2/weib1/kurt/unif/pwr2unif;
Kpower = A(3,:);                % Adjust 1st index acc to sample size
clear A;
Klevel = .01:.01:.20;           % Vector of corresonding sig levels

   % Data for the Power for the 2-sided Skewness Test

load ../../../powerstudy2/weib1/skew/unif/pwr2unif;
SKpower = A(3,:);                % Adjust 1st index acc to sample size
clear A;

plot(sLevel, sPower,Klevel, Kpower,'gx',Klevel, SKpower,'ro',Klevel,
     Kpower,'g',Klevel, SKpower,'r')
title('Sample size = 15')
xlabel('Significance Level')
ylabel('Power')
%legend('Sequential Test','Kurtosis Test','Skewness Test',4)

% ------------------------------- Sample size 25 -----------------------

   % Get Data for Power of the Sequential Test

subplot(2,2,3)
load ../../../shape1/sigtable1;
Level=A(:,:,5);                 % Load the sig levels of the sample size
clear A;
load ../../../powerstudy/weib1/unif/pwrunif;
Power = A(:,:,5);               % Load the power for the same sample size
clear A;
Power=Power(:);                 % String out the arrays into vectors
Level=Level(:);
[sLevel i] = sort(Level);       % Sort the levels and save the indicies to
sPower = Power(i);              % sort the corresponding powers pairwise

   % Data for the Power for the 2-sided Kurtosis Test

load ../../../powerstudy2/weib1/kurt/unif/pwr2unif;
Kpower = A(5,:);                % Adjust 1st index acc to sample size
clear A;
```

```
Klevel = .01:.01:.20;          % Vector of corresonding sig levels

   % Data for the Power for the 2-sided Skewness Test

load ../../../powerstudy2/weib1/skew/unif/pwr2unif;
SKpower = A(5,:);               % Adjust 1st index acc to sample size
clear A;

plot(sLevel, sPower,Klevel, Kpower,'gx',Klevel, SKpower,'ro',Klevel,
     Kpower,'g',Klevel, SKpower,'r')
title('Sample size = 25')
xlabel('Significance Level')
ylabel('Power')
%legend('Sequential Test','Kurtosis Test','Skewness Test',4)

% ------------------------- Sample Size 50 ---------------------------------

   % Get Data for Power of the Sequential Test

subplot(2,2,4)
load ../../../shape1/sigtable1;
Level=A(:,:,10);                % Load the sig levels of the sample size
clear A;
load ../../../powerstudy/weib1/unif/pwrunif;
Power = A(:,:,10);              % Load the power for the same sample size
clear A;
Power=Power(:);                % String out the arrays into vectors
Level=Level(:);
[sLevel i] = sort(Level);      % Sort the levels and save the indicies to
sPower = Power(i);             % sort the corresponding powers pairwise

   % Data for the Power for the 2-sided Kurtosis Test

load ../../../powerstudy2/weib1/kurt/unif/pwr2unif;
Kpower = A(10,:);               % Adjust 1st index acc to sample size
clear A;
Klevel = .01:.01:.20;          % Vector of corresonding sig levels

   % Data for the Power for the 2-sided Skewness Test

load ../../../powerstudy2/weib1/skew/unif/pwr2unif;
SKpower = A(10,:);               % Adjust 1st index acc to sample size
clear A;

plot(sLevel, sPower,Klevel, Kpower,'gx',Klevel, SKpower,'ro',Klevel,
     Kpower,'g',Klevel, SKpower,'r')
title('Sample size = 50')
xlabel('Significance Level')
ylabel('Power')
legend('Sequential Test','Kurtosis Test','Skewness Test',4)
```

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Vita

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13. ABSTRACT *(Maximum 200 words)*
The Weibull distribution finds wide applicability across a broad spectrum of disciplines and is very prevalent in reliability theory. Consequently, numerous statistical tests have been developed to determine whether sample data can be adequately modeled with this distribution. Unfortunately, the majority of these goodness-of-fit tests involve a substantial degree of computational complexity. The study presented here develops and evaluates a new sequential goodness-of-fit test for the three-parameter Weibull distribution with a known shape that delivers power comparable to popular procedures while dramatically reducing computational requirements. The new procedure consists of two distinct tests, using only the sample skewness and sample kurtosis as test statistics. Critical values are derived using large Monte Carlo simulations for known shapes k=0.5(0.5)4 and sample sizes n=5(5)50. Attained significance levels for all combinations of the two tests between alpha=0.01(0.01)0.20 are also are approximated with Monte Carlo simulations and presented in a useful contour plot format. Extensive power studies against numerous alternate distributions demonstrate the test's excellent performance compared to popular EDF test statistics such as the Anderson-Darling and Cramer-von Mises tests. Recommendations are included on techniques to chose significance levels of the two component tests in a manner that should optimize power while maintaining the overall significance level

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